Published by Institute of Physics Publishing for SISSA

Received: *October 2, 2007*
Accepted: *October 26, 2007*
Published: *November 13, 2007*

Large winding sector of AdS/CFT

Hirotaka Hayashi,[a] **Keisuke Okamura,**[ab] **Ryo Suzuki**[a] **and Benoît Vicedo**[b]

[a] *Department of Physics, Faculty of Science, University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan*
[b] *DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, U.K.*
E-mail: hirotakahayashi@hep-th.phys.s.u-tokyo.ac.jp, okamura@hep-th.phys.s.u-tokyo.ac.jp, ryo@hep-th.phys.s.u-tokyo.ac.jp, B.Vicedo@damtp.cam.ac.uk

Abstract: We study a family of classical strings on $\mathbb{R} \times S^3$ subspace of the $AdS_5 \times S^5$ background that interpolates between pulsating strings and single-spike strings. They are obtained from the helical strings of hep-th/0609026 by interchanging worldsheet time and space coordinates, which maps rotating/spinning string states with large spins to oscillating states with large winding numbers. From a finite-gap perspective, this transformation is realised as an interchange of quasi-momentum and quasi-energy defined for the algebraic curve. The gauge theory duals are also discussed, and are identified with operators in the non-holomorphic sector of $\mathcal{N} = 4$ super Yang-Mills. They can be viewed as excited states above the "antiferromagnetic" state, which is "as far from BPS as possible" in the spin-chain spectrum. Furthermore, we investigate helical strings on $AdS_3 \times S^1$ in an appendix.

Keywords: Bethe Ansatz, Integrable Field Theories, AdS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence [1] claims the type IIB string theory on $AdS_5 \times S^5$ is a dual description of the four-dimensional, $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. One of the predictions of the AdS/CFT is the exact matching of the spectra on both sides, namely the conformal dimensions of SYM operators with the energies of string states. In the large-N limit, these charges are supposed to be interpolated by some function of the 't Hooft coupling λ, but the strong/weak nature of the AdS/CFT usually prevents us from direct comparison of the spectra.

Nevertheless, there has been considerable progress in matching the spectra recently, based on the integrable structures of both theories. They are captured by Bethe ansatz equations, which were was first applied to the gauge theory side in the pioneering work of [2]. Despite the fact that we are lacking the knowledge of perturbative computations for higher loop orders in λ even for rather simple rank-one sectors, an all-order asymptotic

Bethe ansatz equation was proposed by assuming all-order integrability as well as making use of some sophisticated guesses [3 – 6]. There has been increasing evidence and positive support for the conjectured Bethe ansatz equation [7 – 13], and significant progress has been achieved in formulating the exact AdS/CFT Bethe ansatz equation valid for all regions of λ.

Many tests of the AdS/CFT conjecture in the large-N limit have taken place in the limit where a $\mathrm{U}(1)_{\mathrm{R}}$-charge J_1 and conformal dimensions Δ of the SYM operators become very large. The BMN limit [14] is one such well-established limit. This limit is defined by sending λ to infinity while keeping $\lambda' \equiv \lambda/J^2$ fixed, where $J = J_1+$number of "impurities".

In [15], a different large-spin limit was considered to serve as a new playground for the AdS/CFT. In this limit, both J_1 and Δ go to infinity while the difference $\Delta - J_1$ and the coupling λ are kept finite. The worldsheet quantum corrections drop out in this limit, which simplifies the comparison of both spectra considerably. Giant magnons are string solutions living in this sector, which have an infinite spin along one of great circles of S^5. They are open objects, and the angular difference between the two endpoints on the equator, which is equal to the localized worldsheet momentum, is identified with the momentum of an excitation in the asymptotic SYM spin-chain.

Giant magnons were generalized to the two-spin case in [16] which carry an additional (finite) second spin J_2, and are know as dyonic giant magnons. In static gauge, the string equations of motion are essentially those of a bosonic $O(4)$ sigma model supplemented by the Virasoro constraints, which is classically equivalent to the Complex sine-Gordon (CsG) system. Thus by using the Pohlmeyer-Lund-Regge (PLR) reduction procedure, the dyonic giant magnon can be constructed as the counterpart of a kink soliton solution of the CsG equation. In this connection, an "elementary" giant magnon of [15] corresponds to a kink soliton of the sine-Gordon (sG) equation. The SYM dual of the dyonic giant magnon is a magnon boundstate in the asymptotic spin-chain [17, 18], where the number of constituent magnons corresponds to the second spin J_2 of the string. It was shown that, in the large-λ limit, the conjectured AdS/CFT S-matrix for boundstates precisely agree with the semiclassical S-matrix for scattering of dyonic giant magnons under an appropriate choice of gauge [19]. For further literature on giant magnons, see [22, 28, 29] (See also [23 – 27]). The idea of exploiting the relation between the classical CsG system and the $O(4)$ string sigma model was further utilized to construct more general classical strings, which are called helical strings [28]. They are the most general "elliptic" classical string solutions on $\mathbb{R} \times S^3$ that interpolate between two-spin folded/circular strings [30] and dyonic giant magnons.

In the algebro-geometric approach to the string equations of motion, these classical string solutions were studied as finite-gap solutions. This line of approach stemmed from the work [31], and has provided many important implications and applications in testing/formulating the conjectured AdS/CFT S-matrix, including the quantum correction [32, 33]. In this formalism, every string solution is characterized by a spectral curve endowed with an Abelian integral called quasimomentum. Recently helical strings were also reconstructed in this framework [34] (see also [35]). It enabled us, in particular, to understand how folded/circular strings and dyonic giant magnons interpolate from the

JHEP11(2007)033

standpoint of algebraic curves.

In this paper, we investigate classical strings on an $\mathbb{R} \times S^3$ subspace of $AdS_5 \times S^5$ with large winding numbers, rather than large spins. The recently found single-spike solution of [36, 37] also falls into this category. In conformal gauge, they are obtained by performing a transformation $\tau \leftrightarrow \sigma$ of large spin states, i.e., interchanging worldsheet time and space of coordinates. Throughout this paper, we will refer to this transformation as the "$\tau \leftrightarrow \sigma$ transformation", or just "2D transformation". This kind of "2D duality" is well-known in the context of rotating strings and pulsating string solutions, both of which are characterized by the same special Neumann-Rosochatius integrable system [38, 39]. For example, if we write the embedding coordinates of $S^3 \subset \mathbb{R}^4$ as $\xi_j = r_j(\tau, \sigma)\ e^{i\varphi_j(\tau,\sigma)}$ $(j = 1, 2)$ with sigma model constraint $\sum_{j=1}^{2} |\xi_j|^2 = 1$, the rotating strings are obtained from the ansatz $r_j = r_j(\sigma)$ and $\varphi_j = w_j\tau + \alpha_j(\sigma)$ with w_j playing the role of angular velocities, while pulsating strings follow from the ansatz $r_j = r_j(\tau)$ and $\varphi_j = m_j\sigma + \alpha_j(\tau)$ with m_j now representing the integer winding numbers. It is reminiscent of T-duality that the angular momenta (spins) and winding numbers are interchanged, however, one should also take notice that not only the angular part φ_j but also the radial part r_j are transformed in our case. To summarize, there are two consequences of this $\tau \leftrightarrow \sigma$ map:

- Large spin states become large winding states.
- Rotating/spinning states become oscillating states.

We will see these features for the case of 2D-transformed helical strings, and see how they interpolate between particular pulsating strings ($\tau \leftrightarrow \sigma$ transformed folded/circular strings) and the single-spike strings ($\tau \leftrightarrow \sigma$ transformed dyonic giant magnons).

It will be also shown that the two classes of string solutions — rotating/spinning with large-spins on the one hand, and oscillating strings with large windings on the other — correspond to two equivalence classes of representations of a generic algebraic curve with two cuts. The $\tau \leftrightarrow \sigma$ operation turns out to correspond to rearranging the configuration of cuts with respect to two singular points on the real axis of the spectral parameter plane.[1]

Concerning the string/spin-chain correspondence of AdS/CFT, we will claim that the dual operators of large-winding oscillating strings are only found in a non-holomorphic sector. Such a non-holomorphic sector has been much less explored than the holomorphic, large-spin sectors, because of its intractability mainly related with the non-closedness, or difficulty of perturbative computations. Nevertheless, since our results, together with the previous works [28, 34], seem to complete the whole catalog of classical, elliptic strings on $\mathbb{R} \times S^3$, we hope they could shed more light not only on holomorphic but also non-holomorphic sectors of the string/spin-chain duality, for a deeper understanding of AdS/CFT. As a first step, in section 5, we will identify the gauge theory duals of the 2D transformed strings.

This paper is organized as follows. In section 2, we briefly review the reduction of classical strings on $\mathbb{R} \times S^3$ to the CsG system, and see the relation between helical strings

[1] An alternative description of $\tau \leftrightarrow \sigma$ operation is to swap the definition of quasi-momentum and so-called quasi-energy. We will make this point clear later in section 4.

JHEP11(2007)033

of [28] and their 2D transformed version from the CsG point of view. In section 3, we study 2D transformed versions of the type (i) and type (ii) strings. These new helical strings are interpreted as finite-gap solutions in section 4. In section 5, we discuss the gauge theory interpretation of the 2D transformed helical strings, and interpret them as excitations above the "antiferromagnetic" state of the SO(6) spin-chain. section 6 is devoted to a summary and discussions. In appendix A, we present similar helical solutions on $AdS_3 \times S^1$. Some computational details useful in discussing the infinite-winding limit can be found in appendix B.

2. 2D-transforming classical strings on $\mathbb{R} \times S^3$

We start with a brief review on how classical strings on $\mathbb{R} \times S^3$ are related to CsG system via the Pohlmeyer-Lund-Regge (PLR) reduction procedure [40], by summarizing the facts in [28].[2] Then we see how the $\tau \leftrightarrow \sigma$ operation acts on the map.

Let us write the metric on $\mathbb{R} \times S^3$ as

$$ds^2_{\mathbb{R}\times S^3} = -d\eta_0^2 + |d\xi_1|^2 + |d\xi_2|^2 \ . \tag{2.1}$$

Here η_0 is the AdS time, and the complex coordinates ξ_j $(j = 1, 2)$ are defined by the embedding coordinates $X_{M=1,\ldots,4}$ of $S^3 \subset \mathbb{R}^4$ as

$$\xi_1 = X_1 + iX_2 = \cos\theta \, e^{i\varphi_1} \qquad \text{and} \qquad \xi_2 = X_3 + iX_4 = \sin\theta \, e^{i\varphi_2} \, . \tag{2.2}$$

We set the radius of S^3 to unity so that $\sum_{M=1}^4 X_M^2 = \sum_{j=1}^2 |\xi_j|^2 = 1$. The Polyakov action for a string which stays at the center of the AdS_5 and rotating on S^3 takes the form,

$$S_{\mathbb{R}\times S^3} = -\frac{\sqrt{\lambda}}{2} \int d\tau \int \frac{d\sigma}{2\pi} \left\{ \gamma^{ab} \left[-\partial_a\eta_0 \, \partial_b\eta_0 + \partial_a\vec{\xi} \cdot \partial_b\vec{\xi}^* \right] + \Lambda(|\vec{\xi}|^2 - 1) \right\} , \tag{2.3}$$

where we used the AdS/CFT relation $\alpha' = 1/\sqrt{\lambda}$, and Λ is a Lagrange multiplier. We take the standard conformal gauge, $\gamma^{\tau\tau} = -1$, $\gamma^{\sigma\sigma} = 1$ and $\gamma^{\sigma\tau} = \gamma^{\tau\sigma} = 0$. Denoting the energy-momentum tensor which follows from the action (2.3) as $\mathcal{T}_{ab}$, the Virasoro constraints are imposed as

$$\begin{aligned} 0 = \mathcal{T}_{\sigma\sigma} = \mathcal{T}_{\tau\tau} &= -\frac{1}{2}(\partial_\tau\eta_0)^2 - \frac{1}{2}(\partial_\sigma\eta_0)^2 + \frac{1}{2}|\partial_\tau\vec{\xi}|^2 + \frac{1}{2}|\partial_\sigma\vec{\xi}|^2 \, , \\ \text{and} \qquad 0 = \mathcal{T}_{\tau\sigma} = \mathcal{T}_{\sigma\tau} &= \mathrm{Re}\left(\partial_\tau\vec{\xi} \cdot \partial_\sigma\vec{\xi}^*\right) . \end{aligned} \tag{2.4}$$

The equations of motion that follow from (2.3) are

$$\partial_a\partial^a\eta_0 = 0 \quad \text{and} \quad \partial_a\partial^a\vec{\xi} + (\partial_a\vec{\xi} \cdot \partial^a\vec{\xi}^*)\vec{\xi} = \vec{0} \, . \tag{2.5}$$

It is well-known that the $O(4)$ (resp. $O(3)$) string sigma model in conformal gauge is classically equivalent to Complex sine-Gordon (resp. sine-Gordon) model with Virasoro constraints [40] (see also [41]). The CsG system is defined by the Lagrangian

$$\mathcal{L}_{\mathrm{CsG}} = \frac{\partial_a\psi^* \, \partial^a\psi}{1 - \psi^*\psi} + \psi^*\psi \, . \tag{2.6}$$

[2] The notation used in this section basically follows from [28].

JHEP11(2007)033

The equation of motion (Complex sine-Gordon equation) which follows from (2.6) is

$$\partial_a \partial^a \psi + \psi^* \frac{(\partial_a \psi)^2}{1-\psi^* \psi} - \psi \left(1 - \psi^* \psi\right) = 0\,. \tag{2.7}$$

The PLR reduction relates the potential term $\partial_a \vec{\xi} \cdot \partial^a \vec{\xi}^*$ with a solution of CsG equation $\psi \equiv \sin\left(\phi/2\right) \exp\left(i\chi/2\right)$, as

$$\partial_a \vec{\xi} \cdot \partial^a \vec{\xi}^* = \cos\phi\,, \tag{2.8}$$

and for each ϕ, one can obtain a consistent classical string solution by solving a Schrödinger type differential equation under appropriate boundary conditions.[3] For example, let us consider a kink soliton solution of CsG equation,

$$\psi(t,x) = \frac{\cos\alpha}{\cosh(x_v \cos\alpha)} \exp\left(it_v \sin\alpha\right)\,, \tag{2.9}$$

where (t_v, x_v) are Lorentz-boosted coordinates

$$t_v \equiv \frac{t - vx}{\sqrt{1-v^2}}\,, \qquad x_v \equiv \frac{x - vt}{\sqrt{1-v^2}}\,. \tag{2.10}$$

Plugging (2.9) into (2.8), and imposing the boundary condition

$$\xi_1 \to \exp\left(it \pm \Delta\varphi_1/2\right), \quad \xi_2 \to 0, \qquad (\text{as } x \to \pm\infty)\,, \tag{2.11}$$

one reaches a dyonic giant magnon [16]. In this case, the angular difference of two endpoints of the string $\Delta\varphi_1$ is determined through the CsG kink parameters α and v.

We are interested in how the 2D transformation acts on the dictionary. Let us first look at the string equations of motion (2.5) and the Virasoro constraints (2.4). In view that they are invariant under the $\tau \leftrightarrow \sigma$ flip, any string solution is mapped to another solution under this map. On closer inspection of the Virasoro constraints (2.4), one actually finds that the $\tau \leftrightarrow \sigma$ operation can be applied independently to the $\mathbb{R} \subset AdS_5$ and $S^3 \subset S^5$ parts. We will use this observation to generate new string solutions from known solutions on $\mathbb{R} \times S^3$, by transforming only the S^3 part while retaining the gauge $t \propto \tau$. In order to satisfy other consistency conditions such as closedness of the string, one needs to care about the periodicity in the new σ direction (that used to be the τ direction before the flip).

Before discussing the CsG counterparts of such $\tau \leftrightarrow \sigma$ transformed string solutions, it would be useful to review some relevant aspects of the (C)sG $\leftrightarrow$ string correspondence before the transformation. A good starting point is the single-spin helical string constructed in [28]. It is a family of classical string on $\mathbb{R} \times S^2$ that interpolates between a folded/circular string of [42] and a giant magnon. From the standpoint of sG theory, the helical string corresponds to the following helical wave ("kink-train") solution of sG equation,

$$\phi(t,x) = 2 \arcsin\left[\operatorname{cn}\left(\frac{(x-x_0) - v(t-t_0)}{k\sqrt{1-v^2}}, k\right)\right]. \tag{2.12}$$

[3]One can also trace back the PLR reduction procedure to obtain CsG solutions from classical string solutions.

via the PLR procedure. The single-spin helical string thus has two controllable parameters derived from the sG soliton (2.12) ; one is the soliton velocity v and the other is the elliptic moduli parameter k that controls the period of the kink-array. In the $k \to 1$ limit, it reduces to an array of giant magnons, while as $v \to 0$, it reduces to a folded/circular string of [42].

Actually there is another periodic solution of sG equation, namely a periodic instanton. Generally, one can interpret a static, finite energy classical solution of sG theory in $(1+1)$-dimensions as a finite action Euclidean solution in $(1+0)$-dimension that interpolates between different vacua of the theory. Such a sG instanton solution is known in the literature (see, e.g., [43]) and is given by

$$\phi(t') = 2 \arcsin \left[\mathrm{cn} \left(\frac{t' - t'_0}{k}, k \right) \right] . \tag{2.13}$$

Here $t' = it$ is the Euclidean time. One can see that a static kink soliton of sG equation $-\partial_x^2 \phi = \sin \phi$ (set $v = 0$ in (2.12)) is related to the instanton (2.13) of the Euclidean sG equation $\partial_{it}^2 \phi = -\partial_{t'}^2 \phi = \sin \phi$ by a formal translation $x \leftrightarrow t'$ (i.e., space-like motion turns into "time-like" motion), which amounts to swapping worldsheet variables $\tau \leftrightarrow \sigma$. Starting from the instanton solution (2.13) , and boosting it by a parameter v , we obtain a one parameter family of sG solutions of the form

$$\phi(t', x') = 2 \arcsin \left[\mathrm{cn} \left(\frac{(t' - t'_0) - v(x' - x'_0)}{k\sqrt{1 - v^2}}, k \right) \right] \tag{2.14}$$

with $(t', x') = (it, ix)$, which is related to the sG helical wave (2.12) by $\tau \leftrightarrow \sigma$.

Via the PLR map, each periodic instanton corresponds to a point-like segment, or "string-bit", and an infinite series of such periodic sG instantons (2.13) arrayed in the σ-direction make up the corresponding classical string. Note that for the boosted instanton (2.14), v no longer represents a velocity, rather it should be viewed as a parameter that controls the difference between time-origins t'_0 for each bits. A pulsating string corresponds to the $v = 0$ case, when the timing of the pulsation of each string-bits is perfectly right. When the pulsation timing of the bits is off in a coherent manner, a symmetric "spike" comes into being, reflecting the staggered motions of bits.[4] In the limit $k \to 1$, the oscillation period of each bit becomes infinite, and the bits stay in the vicinity of the equator for an infinite amount of time, except during a short sudden jump away from the equator — this is one way to interpret the single-spin single-spike string of [36] from the sG point of view.[5]

We have just discussed the way to realise the oscillating solutions resulting from a $\tau \leftrightarrow \sigma$ transformation in terms of a collection of sG instantons. We gave this interpretation

[4] The situation is much the same as the case of familiar transverse waves, where oscillation in the medium takes place in a perpendicular direction to its own motion. This direction of motion corresponds to, in our case, the circumferential direction along the equator of the sphere.

[5] As is noticed in [36], for sG case, it is also possible to argue that the $\tau \leftrightarrow \sigma$ transformation results in the change of sG kink soliton from $\phi = 2 \arcsin (1/\cosh x_v)$ to $\phi = 2 \arcsin (\tanh x_v)$. However, it seems this interpretation cannot be directly applied to CsG case.

because it is very intuitive. Actually one cannot generalise this argument to the CsG case directly, since in this case the argument requires χ to be imaginary. So for the CsG case, it would be convenient instead to interpret the effect of the $\tau \leftrightarrow \sigma$ operation as flipping the sign of the "mass" term in the Lagrangian as

$$\mathcal{L}_{\rm CsG} = \frac{\partial_a \psi^* \, \partial^a \psi}{1 - \psi^* \psi} + \psi^* \psi \quad \mapsto \quad \frac{\partial_a \psi^* \, \partial^a \psi}{1 - \psi^* \psi} - \psi^* \psi \, .$$

In this way one can easily understand how one solution of CsG is related to another via the $\tau \leftrightarrow \sigma$ transformation (keeping ϕ and χ real).

Notice also, as in the soliton cases, that there are two classes of "boosted" instantons possible; the first is an instanton that oscillates about one of the barriers of the periodic potential with fixed finite oscillation range, while the other no longer oscillates back and forth but goes on from one barrier to the neighboring one. A similar kind of distinction exists for what we call type $(i)'$ and type $(ii)'$ strings.

3. Helical oscillating strings

We are now in a position to discuss the 2D transformed helical strings. We first study the type $(i)'$ case in the following section 3.1. The results on the type $(ii)'$ solutions will be collected in section 3.2.

3.1 Type $(i)'$ helical strings

For the reader's convenience, let us display the profile of the two-spin helical string obtained in [28],[6]

$$\eta_0^{\rm orig} = aT + bX \, , \tag{3.1}$$

$$\xi_1^{\rm orig} = C \frac{\Theta_0(0)}{\sqrt{k}\, \Theta_0(i\omega_1)} \frac{\Theta_1(X - i\omega_1)}{\Theta_0(X)} \exp\Big(Z_0(i\omega_1) X + i u_1 T \Big) \, , \tag{3.2}$$

$$\xi_2^{\rm orig} = C \frac{\Theta_0(0)}{\sqrt{k}\, \Theta_2(i\omega_2)} \frac{\Theta_3(X - i\omega_2)}{\Theta_0(X)} \exp\Big(Z_2(i\omega_2) X + i u_2 T \Big) \, , \tag{3.3}$$

where ω_1 and ω_2 are real parameters, k is the elliptic modulus, and C is the normalization constant given by

$$C = \left(\frac{{\rm dn}^2(i\omega_2)}{k^2 \, {\rm cn}^2(i\omega_2)} - {\rm sn}^2(i\omega_1) \right)^{-1/2} \, . \tag{3.4}$$

The coordinates (T, X) are defined by

$$T = \frac{\tilde{\tau} - v\tilde{\sigma}}{\sqrt{1 - v^2}} \, , \quad X = \frac{\tilde{\sigma} - v\tilde{\tau}}{\sqrt{1 - v^2}} \, , \qquad (\tilde{\tau}, \tilde{\sigma}) \equiv (\mu\tau, \mu\sigma) \tag{3.5}$$

[6]Throughout this paper, we often omit the elliptic moduli k from expressions of elliptic functions. For example, we will often write $\Theta_\nu(z)$ or $\mathbf{K}$ instead of $\Theta_\nu(z, k)$ or $\mathbf{K}(k)$.

with μ constant. Starting from (3.1)–(3.3), by swapping τ and σ in $\xi_i(\tau,\sigma)$ $(i=1,2)$ while keeping the relation $\eta_0(\tau,\sigma)=aT+bX$ as it is, one obtains the 2D-transformed version of the type (i) two-spin helical strings, which we call type $(i)'$ helical strings,

$$\xi_1 = C\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_0(i\omega_1)}\frac{\Theta_1(T-i\omega_1)}{\Theta_0(T)}\,\exp\Big(Z_0(i\omega_1)T+iu_1X\Big)\,, \tag{3.6}$$

$$\xi_2 = C\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_2(i\omega_2)}\frac{\Theta_3(T-i\omega_2)}{\Theta_0(T)}\,\exp\Big(Z_2(i\omega_2)T+iu_2X\Big)\,. \tag{3.7}$$

The Virasoro constraints (2.4) fix the parameters a and b in (3.1),

$$a^2+b^2 = k^2-2k^2\,\mathrm{sn}^2(i\omega_1)-U+2u_2^2\,, \tag{3.8}$$

$$ab = -i\,C^2\left(u_1\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)-u_2\,\frac{1-k^2}{k^2}\,\frac{\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)}\right)\,. \tag{3.9}$$

We can adjust the parameter v such that the AdS time is proportional to the worldsheet time variable, namely $\eta_0=\sqrt{a^2-b^2}\,\tilde{\tau}$ with $v\equiv b/a\le 1$. The PLR reduction relation (2.8) becomes

$$\frac{1}{\mu^2}\sum_{i=1}^{2}\Big(|\partial_\sigma\xi_i|^2-|\partial_\tau\xi_i|^2\Big) = -k^2+2k^2\,\mathrm{sn}^2(T)+U\,, \tag{3.10}$$

which imposes the following constraints among the parameters

$$u_1^2 = U+\mathrm{dn}^2(i\omega_1)\,,\qquad u_2^2 = U-\frac{(1-k^2)\,\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)}\,. \tag{3.11}$$

We are interested in closed string solutions, which means we need to consider the periodicity conditions. The period in σ-direction is defined such that it leaves the theta functions in (3.2) and (3.3) invariant, namely it is given by

$$-\ell\le\sigma\le\ell,\qquad \ell=\frac{\mathbf{K}\sqrt{1-v^2}}{v\mu}\,,\qquad (v>0)\,. \tag{3.12}$$

Then, closedness of the string requires

$$\Delta\sigma \equiv \frac{2\pi}{n} = \frac{2\mathbf{K}\sqrt{1-v^2}}{v\mu}\,, \tag{3.13}$$

$$\Delta\varphi_1 \equiv \frac{2\pi N_1}{n} = 2\mathbf{K}\left(\frac{u_1}{v}+iZ_0(i\omega_1)\right)+(2n_1'+1)\pi\,, \tag{3.14}$$

$$\Delta\varphi_2 \equiv \frac{2\pi N_2}{n} = 2\mathbf{K}\left(\frac{u_2}{v}+iZ_2(i\omega_2)\right)+2n_2'\pi\,, \tag{3.15}$$

where $n=1,2,\ldots$ counts the number of periods in $0\le\sigma\le 2\pi$, and $N_{1,2}$ are the winding numbers in $\varphi_{1,2}$-directions respectively. The integers $n_{1,2}'$ specify the ranges of $\omega_{1,2}$ respectively (the shifts $\omega_i\mapsto\omega_i+2\mathbf{K}'$ correspond to $n_i'\mapsto n_i'+1$).

The energy $E=(\sqrt{\lambda}/\pi)\,\mathcal{E}$ and spins $J_i=(\sqrt{\lambda}/\pi)\,\mathcal{J}_i$ $(i=1,2)$ of the string with n periods are obtained from the usual definitions

$$\mathcal{E}=\int_{-n\ell}^{n\ell}d\sigma\,\partial_\tau\eta_0\,,\qquad \mathcal{J}_i=\frac{1}{2}\int_{-n\ell}^{n\ell}d\sigma\,\mathrm{Im}\,(\xi_i^*\partial_\tau\xi_i)\,, \tag{3.16}$$

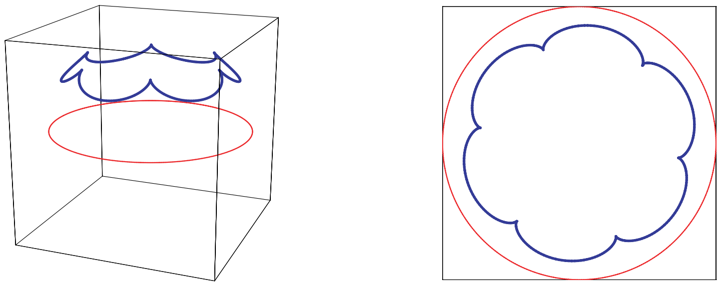

Figure 1: Type $(i)'$ helical string $(k = 0.68\,, n = 6)$, projected onto S^2. The figure shows a single-spin case $(u_2 = \omega_2 = 0)$. The (red) circle indicates the $\theta = 0$ line (referred to as the "equator" in the main text).

which yield in the present case,

$$\mathcal{E} = \frac{na(1-v^2)}{v}\,\mathbf{K} = \frac{n(a^2-b^2)}{b}\,\mathbf{K}\,, \tag{3.17}$$

$$\mathcal{J}_1 = \frac{nC^2\,u_1}{k^2}\left[\mathbf{E} - \left(\mathrm{dn}^2(i\omega_1) + \frac{i\,k^2}{v\,u_1}\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right], \tag{3.18}$$

$$\mathcal{J}_2 = \frac{nC^2\,u_2}{k^2}\left[-\mathbf{E} - (1-k^2)\left(\frac{\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)} - \frac{i}{v\,u_2}\frac{\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)}\right)\mathbf{K}\right]. \tag{3.19}$$

It is meaningful to compare the above expressions with the ones for the original type (i) helical strings of [28],

$$\mathcal{E}^{\mathrm{orig}} = n\,a\left(1-v^2\right)\mathbf{K} = \frac{n(a^2-b^2)}{a}\,\mathbf{K}\,, \tag{3.20}$$

$$\mathcal{J}_1^{\mathrm{orig}} = \frac{nC^2\,u_1}{k^2}\left[-\mathbf{E} + \left(\mathrm{dn}^2(i\omega_1) + \frac{i\,v\,k^2}{u_1}\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right], \tag{3.21}$$

$$\mathcal{J}_2^{\mathrm{orig}} = \frac{nC^2\,u_2}{k^2}\left[\mathbf{E} + (1-k^2)\left(\frac{\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)} - \frac{i\,v}{u_2}\frac{\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)}\right)\mathbf{K}\right]. \tag{3.22}$$

If we regard $\mathcal{E}$ and $\mathcal{J}_i$ as functions of $v = b/a$, the global charges of the transformed solutions are related to the original ones by $\mathcal{E}(a,b) = -\mathcal{E}^{\mathrm{orig}}(b,a)$ and $\mathcal{J}_i(v) = -\mathcal{J}_i^{\mathrm{orig}}(-1/v)$. Similar relations are also true for the winding numbers given in (3.14) and (3.15), $N_i(v) = -N_i^{\mathrm{orig}}(-1/v)$ $(i = 1, 2)$. They are just a consequence of the symmetry $a \leftrightarrow b$ the Virasoro constraints possess. For example, if $(a,b) = (a_0, b_0)$ solves (3.8) and (3.9), then $(a,b) = (b_0, a_0)$ gives another solution.

Notice that in the limit $v \to 0$ $(\omega_{1,2} \to 0)$, all the winding numbers in (3.13)-(3.15) become divergent (and so ill-defined), due to the fact that the θ defined in (2.2) becomes independent of σ. Therefore, in this limiting case, we may choose μ arbitrarily without the need of solving (3.13), provided that N_1 and N_2 are both integers.

The type $(i)'$ helical strings contains both pulsating strings and single-spike strings in particular limits.

$\omega_{1,2} \to 0$ limit: pulsating strings. Let us first consider the $\omega_{1,2} \to 0$ limit. In this limit, the boosted coordinates (3.5) reduce to $(T, X) \to (\tilde{\tau}, \tilde{\sigma})$, and (3.1), (3.6)–(3.7) become

$$\eta_0 = \sqrt{k^2 + u_2^2}\, \tilde{\tau}\,, \qquad \xi_1 = k\, \mathrm{sn}(\tilde{\tau}, k)\, e^{iu_1\tilde{\sigma}}\,, \qquad \xi_2 = \mathrm{dn}(\tilde{\tau}, k)\, e^{iu_2\tilde{\sigma}}\,, \tag{3.23}$$

with the constraint $u_1^2 - u_2^2 = 1$. Since the radial direction is independent of σ, we may treat μ as a free parameter satisfying $N_1 = \mu u_1$ and $N_2 = \mu u_2$. Then the conserved charges for a period become

$$\mathcal{E} = \pi k \sqrt{N_1^2 + \left(\frac{1}{k^2} - 1\right) N_2^2}\,, \qquad \mathcal{J}_1 = \mathcal{J}_2 = 0\,. \tag{3.24}$$

Left of figure 2 shows the time evolution of the type $(i)'$ pulsating string. It stays above the equator, and sweeps back and forth between the pole ($\theta = \frac{\pi}{2}$) and the turning latitude determined by k.

When we set $u_2 = 0$, this string becomes identical to the simplest pulsating solution studied in [44] (the zero-rotation limit of rotating and pulsating strings studied in [45, 46]).[7]

$k \to 1$ limit: single-spike strings. When the moduli parameter k goes to unity, type $(i)'$ helical string becomes an array of single-spike strings studied in [36, 37]. Dependence on ω_2 drops out in this limit, so we write ω instead ω_1. The Virasoro constraints can be explicitly solved by setting $a = u_1$ and $b = \tan\omega$. The profile of the string then becomes

$$\eta_0 = \sqrt{1 + u_2^2}\, \tilde{\tau}\,, \qquad \xi_1 = \frac{\sinh(T - i\omega)}{\cosh(T)}\, e^{i\tan(\omega)T + iu_1 X}\,, \qquad \xi_2 = \frac{\cos(\omega)}{\cosh(T)}\, e^{iu_2 X}\,. \tag{3.25}$$

with the constraint $u_1^2 - u_2^2 = 1 + \tan^2\omega$.[8] The conserved charges are computed as

$$\mathcal{E} = \left(\frac{u_1^2 - \tan^2\omega}{\tan\omega}\right) \mathbf{K}(1)\,, \qquad \mathcal{J}_1 = u_1 \cos^2\omega\,, \qquad \mathcal{J}_2 = u_2 \cos^2\omega\,, \tag{3.26}$$

where $\mathbf{K}(1)$ is a divergent constant. For $n = 1$ case (single spike), the expressions (3.26) result in

$$\mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + \cos^2\omega}\,, \qquad \text{i.e.,} \quad J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\cos^2\omega}\,. \tag{3.27}$$

Since the winding number $\Delta\varphi_1$ also diverges as $k \to 1$, this limit can be referred to as the "infinite winding" limit,[9] which can be viewed as the 2D-transformed version of the infinite spin limit of [15]. By examining the periodicity condition carefully, one finds that

[7] The type $(i)'$ pulsating solution studied here and also the type $(ii)'$ pulsating string discussed later are qualitatively different solutions from the so called "rotating pulsating string" [45], so that the finite-gap interpretation and the gauge theory interpretation of type $(i)'$ and $(ii)'$ are also different from those of [45].

[8] Here $u_{1,2}$ and ω are related to γ used in [36] (see their eq. (6.23)) by $u_1 = \frac{1}{\cos\gamma\cos\omega}$ and $u_2 = \frac{\tan\gamma}{\cos\omega}$.

[9] Notice, however, that the string wraps very close to the equator but touches it only once every period (every "cusp").

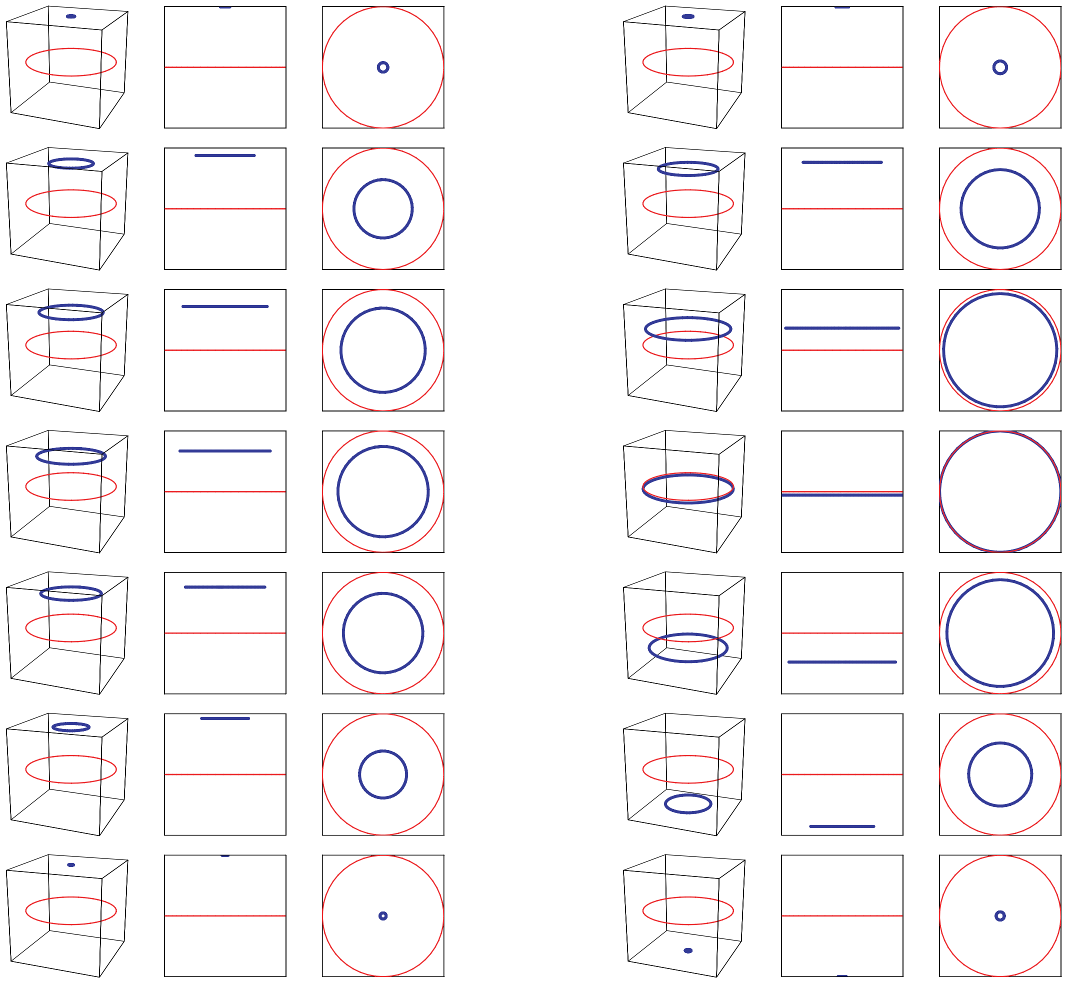

Figure 2: In the $\omega_{1,2} \to 0$ limit, type $(i)'$ (Left figure) and type $(ii)'$ (Right figure) helical strings reduce to different types of pulsating strings. Their behaviors are different in that the type $(i)'$ sweeps back and forth only in the top hemisphere with turning latitude controlled by the elliptic modulus, while the type $(ii)'$ pulsates on the entire sphere, see section 3.2. For the type $(ii)'$ case, we only showed half of the oscillation period (for the other half, it sweeps back from the south pole to the north pole).

both of the divergences come from the same factor $\mathbf{K}(k)|_{k\to 1}$. Using the formula (B.6), one can deduce that

$$\mathcal{E} - \frac{\Delta\varphi_1}{2}\bigg|_{k\to 1} = -\left(\omega - \frac{(2n_1'+1)\,\pi}{2}\right) \equiv \bar{\theta}\,. \tag{3.28}$$

Using the $\bar{\theta}$ variable introduced above, which is the same definition as used in [36], one can see (3.27) precisely reproduces the relation between spins obtained in [36].

Let us comment on a subtly about $v \to 0$ (or equivalently $\omega \to 0$) limit of a single spike string. It is easy to see the profile of single-spike solution (3.25) with $\omega = 0$ agrees with that of pulsating string solution (3.23) with $k = 1$, however, due to a singular nature

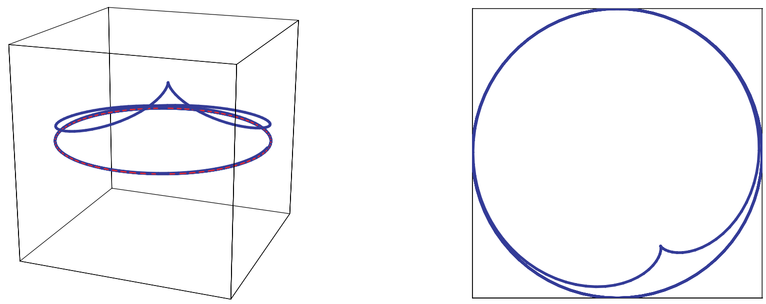

Figure 3: The $k \to 1$ limit of type $(i)'$ helical string: single-spike string $(\omega = 0.78)$. The figure shows the single-spin case $(u_2 = \omega_2 = 0)$.

of the $v \to 0$ limit, the angular momenta of both solutions (3.27) and (3.24) do not agree if we just naively take the limits on both sides.

$k \to 0$ limit: rational circular (static) strings. Another interesting limit is to send k to zero, where elliptic functions reduce to rational functions. The Virasoro conditions become

$$a^2 + b^2 = u_2^2 + \tanh^2 \omega_2 \quad \text{and} \quad ab = \pm u_2 \tanh \omega\,, \tag{3.29}$$

where $u_2 = \sqrt{U + \tanh^2 \omega}$. This can be solved by $a = u_2$ and $b = \tanh \omega$ (assuming $U > 0$). The profile is given by

$$\eta_0 = \sqrt{U}\tilde{\tau}\,, \qquad \xi_1 = 0\,, \qquad \xi_2 = e^{i\sqrt{U}\tilde{\sigma}}\,. \tag{3.30}$$

This is an unstable string that has no spins and just wraps around one of the great circles, and can be viewed as the $\tau \leftrightarrow \sigma$ transformed version of a point-like, BPS string with $E - (J_1 + J_2) = 0$. The conserved charges for one period reduce to

$$\mathcal{E} = \pi\mu\sqrt{U}\,, \qquad \mathcal{J}_1 = \mathcal{J}_2 = 0\,. \tag{3.31}$$

The winding number for the φ_2-direction becomes $N_2 = \mu\sqrt{U}$, so the energy can also be written as

$$E = N_2\sqrt{\lambda}\,. \tag{3.32}$$

This result will be suggestive when we discuss gauge theory later in section 5, since it predicts that the canonical dimension of SYM dual operator, which should be the SO(6) singlet state, is also given by (integer) $\times \sqrt{\lambda}$ in this limit. Note also that in the limit $\mu\sqrt{U} \to \infty$, the profile (3.30) agrees with the $\omega = \pi/2$ case of the single-spike string after the interchange $\xi_1 \leftrightarrow \xi_2$. We will refer to this fact in the gauge theory discussion.

$u_2, \omega_2 \to 0$: single-spin limit. A single-spin type $(i)'$ helical string is obtained by setting $u_2 = \omega_2 = 0$, which results in $J_2 = N_2 = 0$.[10] In view of (3.11), the condition

[10] It turns out the other single-spin limit $u_1, \omega_1 \to 0$, which gives $J_1 = 0$, does not result in real solutions for this type $(i)'$ case.

$u_2 = \omega_2 = 0$ requires $U = 0$, $u_1 = \mathrm{dn}(i\omega)$ and $C = \sqrt{k}/\,\mathrm{dn}(i\omega)$, and the Virasoro constraints (3.8) and (3.9) are solved by setting $a = k\,\mathrm{cn}(i\omega)$, $b = -ik\,\mathrm{sn}(i\omega)$ and $v = -i\,\mathrm{sn}(i\omega)/\,\mathrm{cn}(i\omega)$. Periodicity conditions then become

$$\Delta\sigma = \frac{2\pi}{n} = \frac{2i\mathbf{K}}{\mu\,\mathrm{sn}(i\omega)}\,, \qquad \frac{2\pi N_2}{n} = 0\,, \tag{3.33}$$

$$\Delta\varphi_1 = \frac{2\pi N_1}{n} = 2i\mathbf{K}\left(\frac{\mathrm{cn}(i\omega)\,\mathrm{dn}(i\omega)}{\mathrm{sn}(i\omega)} + Z_0(i\omega)\right) + \left(2n_1' + 1\right)\pi\,, \tag{3.34}$$

and the conserved charges for one period are

$$\mathcal{E} = \frac{ik}{\mathrm{sn}(i\omega)}\mathbf{K}\,, \qquad \mathcal{J}_1 = \frac{1}{k\,\mathrm{dn}(i\omega)}\Big[\mathbf{E} - \left(1-k^2\right)\mathbf{K}\Big]\,, \qquad \mathcal{J}_2 = 0\,. \tag{3.35}$$

3.2 Type $(ii)'$ helical strings

The type $(ii)'$ solution can be obtained from the type $(i)'$ solutions, either by shifting $\omega_2 \mapsto \omega_2 + \mathbf{K}'$ or by transforming k to $1/k$. The profile is given by[11]

$$\hat{\eta}_0 = \hat{a}T + \hat{b}X\,, \tag{3.36}$$

$$\hat{\xi}_1 = \hat{C}\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_0(i\omega_1)}\frac{\Theta_1(T-i\omega_1)}{\Theta_0(T)}\,\exp\Big(Z_0(i\omega_1)T + iu_1X\Big)\,, \tag{3.37}$$

$$\hat{\xi}_2 = \hat{C}\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_3(i\omega_2)}\frac{\Theta_2(T-i\omega_2)}{\Theta_0(T)}\,\exp\Big(Z_3(i\omega_2)T + iu_2X\Big)\,, \tag{3.38}$$

where $\hat{C}$ is the normalization constant,

$$\hat{C} = \left(\frac{\mathrm{cn}^2(i\omega_2)}{\mathrm{dn}^2(i\omega_2)} - \mathrm{sn}^2(i\omega_1)\right)^{-1/2}. \tag{3.39}$$

The equations of motion force u_1 and u_2 to satisfy

$$u_1^2 = U + \mathrm{dn}^2(i\omega_1)\,, \qquad u_2^2 = U + \frac{1-k^2}{\mathrm{dn}^2(i\omega_2)}\,, \tag{3.40}$$

and the Virasoro conditions impose the following constraints between parameters $\hat{a}$ and $\hat{b}$,

$$\hat{a}^2 + \hat{b}^2 = k^2 - 2k^2\,\mathrm{sn}^2(i\omega_1) - U + 2u_2^2\,, \tag{3.41}$$

$$\hat{a}\,\hat{b} = -i\,\hat{C}^2\left(u_1\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1) + u_2\left(1-k^2\right)\frac{\mathrm{sn}(i\omega_2)\,\mathrm{cn}(i\omega_2)}{\mathrm{dn}^3(i\omega_2)}\right). \tag{3.42}$$

As in the type $(i)'$ case, we can set $\hat{\eta}_0 = \sqrt{\hat{a}^2 - \hat{b}^2}\,\tilde{\tau}$ with $\hat{v} \equiv \hat{b}/\hat{a} \le 1$. The periodicity conditions for the type $(ii)'$ solutions become

$$\Delta\sigma \equiv \frac{2\pi}{m} = \frac{2\mathbf{K}\sqrt{1-\hat{v}^2}}{\hat{v}\,\mu}\,, \tag{3.43}$$

$$\Delta\varphi_1 \equiv \frac{2\pi M_1}{m} = 2\mathbf{K}\left(\frac{u_1}{\hat{v}} + iZ_0(i\omega_1)\right) + \left(2m_1' + 1\right)\pi\,, \tag{3.44}$$

$$\Delta\varphi_2 \equiv \frac{2\pi M_2}{m} = 2\mathbf{K}\left(\frac{u_2}{\hat{v}} + iZ_3(i\omega_2)\right) + \left(2m_2' + 1\right)\pi\,, \tag{3.45}$$

[11]We use a hat to indicate type $(ii)'$ variables.

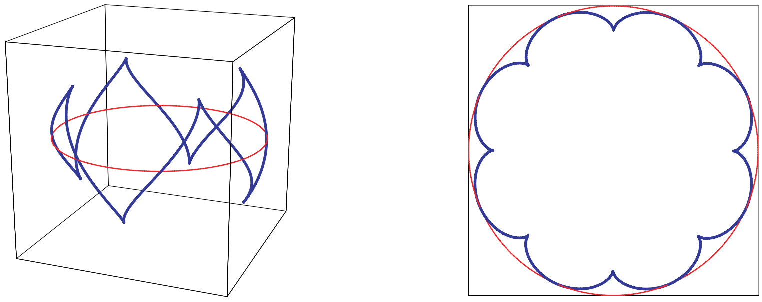

Figure 4: Type $(ii)'$ helical string ($k = 0.40\,, m = 8$). The figure shows a single-spin case ($u_2 = \omega_2 = 0$).

where $m = 1, 2, \ldots$ counts the number of periods in $0 \le \sigma \le 2\pi$, and $M_{1,2}$ are the winding numbers in the $\varphi_{1,2}$-directions respectively, and $m'_{1,2}$ are integers. The conserved charges are given by

$$\hat{\mathcal{E}} = \frac{ma(1-v^2)}{v}\,\mathbf{K} = \frac{n(a^2-b^2)}{b}\,\mathbf{K}\,, \tag{3.46}$$

$$\hat{\mathcal{J}}_1 = \frac{m\hat{C}^2\,u_1}{k^2}\left[\mathbf{E} - \left(\mathrm{dn}^2(i\omega_1) + \frac{i\,k^2}{\hat{v}\,u_1}\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right]\,, \tag{3.47}$$

$$\hat{\mathcal{J}}_2 = \frac{m\hat{C}^2\,u_2}{k^2}\left[-\mathbf{E} + (1-k^2)\left(\frac{1}{\mathrm{dn}^2(i\omega_2)} - \frac{i\,k^2}{\hat{v}\,u_2}\,\frac{\mathrm{sn}(i\omega_2)\,\mathrm{cn}(i\omega_2)}{\mathrm{dn}^3(i\omega_2)}\right)\mathbf{K}\right]\,. \tag{3.48}$$

Just as in the type $(i) \leftrightarrow (i)'$ case, the winding numbers and the conserved charges of the original type (ii) and $(ii)'$ are related by $\hat{\mathcal{E}}(\hat{a},\hat{b}) = -\hat{\mathcal{E}}^{\mathrm{orig}}(\hat{b},\hat{a})$, $\hat{\mathcal{J}}_i(\hat{v}) = -\hat{\mathcal{J}}_i^{\mathrm{orig}}(-1/\hat{v})$ and $M_i(\hat{v}) = -M_i^{\mathrm{orig}}(-1/\hat{v})$.

As in the type $(i)'$ case, we can take various limits.

$\omega_{1,2} \to 0$ limit: pulsating strings. The profiles (3.36)-(3.38) reduce to

$$\hat{\eta}_0 = \sqrt{1+u_2^2}\;\tilde{\tau}\,, \qquad \hat{\xi}_1 = \mathrm{sn}(\tilde{\tau},k)\,e^{iu_1\tilde{\sigma}}\,, \qquad \hat{\xi}_2 = \mathrm{cn}(\tilde{\tau},k)\,e^{iu_2\tilde{\sigma}}\,, \tag{3.49}$$

with constraint $u_1^2 - u_2^2 = k^2$. The conserved charges for a period become

$$\mathcal{E} = \frac{\pi}{k}\sqrt{M_1^2 + (k^2-1)\,M_2^2}\,, \qquad \mathcal{J}_1 = \mathcal{J}_2 = 0\,. \tag{3.50}$$

Right of figure 2 shows the time evolution of the type $(ii)'$ pulsating string. Again, when we set $u_2 = 0$, this string reduces to the simplest pulsating solution studied in [44].

$k \to 1$ limit: single-spike strings. This limit results in essentially the same solution as the type $(i)'$ case, that is an array of single-spike strings. The only difference is that while in the type $(i)'$ case every cusp appears in the same side about the equator, say the northern hemisphere, in the type $(ii)'$ case cusps appear in both the northern and southern hemispheres in turn, each after an infinite winding.

$k \to 0$ limit: rational circular strings. In the $k \to 0$ limit, the profile becomes

$$\hat{\eta}_0 = \sqrt{\hat{a}^2 - \hat{b}^2}\,\tilde{\tau}\,, \quad \hat{\xi}_1 = \hat{C}\;\sin(T - i\omega_1)\,e^{iu_1 X}\,, \quad \hat{\xi}_2 = \hat{C}\;\cos(T - i\omega_2)\,e^{iu_2 X}\,, \tag{3.51}$$

with $\hat{C} = \left(\cosh^2\omega_2 + \sinh^2\omega_1\right)^{-1/2}$ and $u_1^2 = u_2^2 = U + 1$. Virasoro constraints imply the following set of relations between the parameters $\hat{a}$ and $\hat{b}$ (with $\hat{a} \geq \hat{b}$):

$$\hat{a}^2 + \hat{b}^2 = -U + 2u_2^2\,, \tag{3.52}$$

$$\hat{a}\,\hat{b} = \hat{C}^2\sqrt{U+1}\,(\sinh\omega_1\cosh\omega_1 \mp \sinh\omega_2\cosh\omega_2)\,. \tag{3.53}$$

Here $\mp$ reflects the sign ambiguity in the angular momenta. The periodicity conditions become

$$\Delta\sigma \equiv \frac{2\pi}{m} = \frac{\pi\sqrt{1-\hat{v}^2}}{\hat{v}\,\mu}\,, \tag{3.54}$$

$$\Delta\varphi_1 \equiv \frac{2\pi M_1}{m} = \frac{\pi\,u_1}{\hat{v}} + \left(2m_1' + 1\right)\pi, \tag{3.55}$$

$$\Delta\varphi_2 \equiv \frac{2\pi M_2}{m} = \frac{\pi\,u_2}{\hat{v}} + \left(2m_2' + 1\right)\pi\,. \tag{3.56}$$

The conserved charges for a single period are evaluated as

$$\hat{\mathcal{E}} = \frac{\pi\hat{a}\left(1-\hat{v}^2\right)}{2\hat{v}}\,, \qquad \hat{\mathcal{J}}_1 = \frac{\pi\hat{C}^2}{2\hat{v}}\sinh\omega_1\cosh\omega_1\,, \qquad \hat{\mathcal{J}}_2 = -\frac{\pi\hat{C}^2}{2\hat{v}}\sinh\omega_2\cosh\omega_2\,. \tag{3.57}$$

$u_2\,, \omega_2 \to 0$: single-spin limit. As in the type $(i)'$ case, we obtain the type $(ii)'$ helical strings with $J_2 = M_2 = 0$ by setting $u_2 = \omega_2 = 0$.[12] Then we find $U = -1 + k^2$, $u_1 = k\,\mathrm{cn}(i\omega)$ and $\hat{C} = 1/\,\mathrm{cn}(i\omega)$. The Virasoro conditions require $\hat{a} = \mathrm{dn}(i\omega)$, $\hat{b} = -ik\,\mathrm{sn}(i\omega)$ and $\hat{v} = -ik\,\mathrm{sn}(i\omega)/\,\mathrm{dn}(i\omega)$. The periodicity conditions become

$$\Delta\sigma = \frac{2\pi}{m} = \frac{2i\mathbf{K}}{\mu k\,\mathrm{sn}(i\omega)}\,, \qquad \frac{2\pi M_2}{m} = 0\,, \tag{3.58}$$

$$\Delta\varphi_1 = \frac{2\pi M_1}{m} = 2i\mathbf{K}\left(\frac{\mathrm{cn}(i\omega)\,\mathrm{dn}(i\omega)}{\mathrm{sn}(i\omega)} + Z_0(i\omega)\right) + \left(2m_1' + 1\right)\pi\,, \tag{3.59}$$

and the conserved charges for a single period are given by

$$\hat{\mathcal{E}} = \frac{i}{k\,\mathrm{sn}(i\omega)}\,\mathbf{K}\,, \qquad \hat{\mathcal{J}}_1 = \frac{1}{k\,\mathrm{cn}(i\omega)}\,\mathbf{E}\,, \qquad \hat{\mathcal{J}}_2 = 0\,. \tag{3.60}$$

[12] For the type $(ii)'$ case, the other single-spin limit $u_1 = \omega_1 = 0$ results in $U = -1$, $u_2^2 = -1 + (1 - k^2)/\,\mathrm{dn}^2(i\omega_2)$ and $\hat{C} = \mathrm{dn}(i\omega_2)/\,\mathrm{cn}(i\omega_2)$. It turns out equivalent to the $\omega_{1,2} \to 0$ limit, because u_2 must be real, and thus the second condition implies $\omega_2 = 0$.

4. Finite-gap interpretation

The helical strings (3.2), (3.3) of [28] were shown in [34] to be equivalent to the most general elliptic ("two-cut") finite-gap solution on $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$, with both cuts intersecting the real axis within the interval $(-1, 1)$ (see figure 5 (a)). The aim of this section is to present the corresponding finite-gap description of the $\tau \leftrightarrow \sigma$ transformed helical string (3.6), (3.7) obtained in the previous section.

Recall first from [34] that the (σ, τ)-dependence of the general finite-gap solution enters solely through the differential form

$$d\mathcal{Q}(\sigma, \tau) = \frac{1}{2\pi} (\sigma dp + \tau dq) , \tag{4.1}$$

where dp and dq are the differentials of the quasi-momentum and quasi-energy defined below by their respective asymptotics near the points $x = \pm 1$. The differential multiplying σ in $d\mathcal{Q}(\sigma, \tau)$ (namely dp) is related to the eigenvalues of the monodromy matrix, which by definition is the parallel transporter along a closed loop $\sigma \in [0, 2\pi]$ on the worldsheet. This is because the Baker-Akhiezer vector $\boldsymbol{\psi}(P, \sigma, \tau)$, whose (σ, τ)-dependence also enters solely through the differential form $d\mathcal{Q}(\sigma, \tau)$ in (4.1), satisfies [35]

$$\boldsymbol{\psi}(P, \sigma + 2\pi, \tau) = \exp\left\{ i \int_{\infty^+}^{P} dp \right\} \boldsymbol{\psi}(P, \sigma, \tau).$$

Now it is clear from (4.1) that the $\sigma \leftrightarrow \tau$ operation can be realised on the general finite-gap solution by simply interchanging the quasi-momentum with the quasi-energy,

$$dp \;\leftrightarrow\; dq \,. \tag{4.2}$$

However, since we wish dp to always denote the differential related to the eigenvalues of the monodromy matrix, by the above argument it must always appear as the coefficient of σ in $d\mathcal{Q}(\sigma, \tau)$. Therefore equation (4.2) should be interpreted as saying that the respective definitions of the differentials dp and dq are interchanged, but $d\mathcal{Q}(\sigma, \tau)$ always takes the same form as in (4.1).

Before proceeding let us recall the precise definitions of these differentials dp and dq. Consider an algebraic curve Σ, which admits a hyperelliptic representation with cuts. For what follows it will be important to specify the position of the different cuts relative to the points $x = \pm 1$, *i.e.*, figures 5 (a) and 5 (b) are to be distinguished for the purpose of defining dp and dq. We could make this distinction by specifying an equivalence relation on representations of Σ in terms of cuts, where two representations are equivalent if the cuts of one can be deformed into the cuts of the other within $\mathbb{C} \setminus \{\pm 1\}$. It is straightforward to see that there are only two such equivalence classes for a general algebraic curve Σ. For example, in the case of an elliptic curve Σ the representatives of these two equivalence classes are given in figures 5 (a) and 5 (b). Now with respect to a given equivalence class of cuts, the differentials dp and dq can be uniquely defined on Σ as in [35] by the following conditions:

- their $\mathcal{A}$-period vanishes.

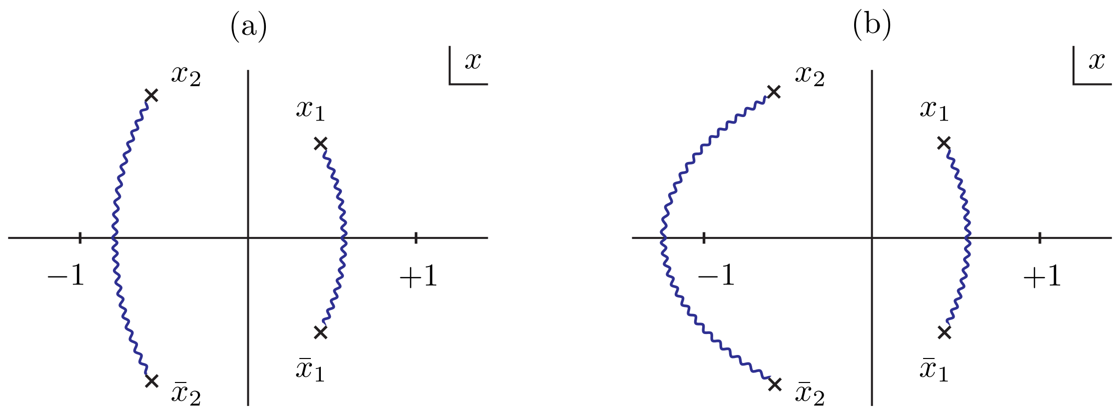

Figure 5: Different possible arrangements of cuts relative to $x = \pm 1$: (a) corresponds to the helical string, (b) corresponds to the $\tau \leftrightarrow \sigma$ transformed helical string.

- their respective poles at $x = \pm 1$ are of the following form, up to a trivial overall change of sign (see [34]),

$$dp(x^\pm) \underset{x \to +1}{\sim} \mp \frac{\pi \kappa dx}{(x-1)^2}, \qquad dp(x^\pm) \underset{x \to -1}{\sim} \mp \frac{\pi \kappa dx}{(x+1)^2}, \tag{4.3}$$

$$dq(x^\pm) \underset{x \to +1}{\sim} \mp \frac{\pi \kappa dx}{(x-1)^2}, \qquad dq(x^\pm) \underset{x \to -1}{\sim} \pm \frac{\pi \kappa dx}{(x+1)^2}, \tag{4.4}$$

  where $x^\pm \in \Sigma$ denotes the pair of points above x, with x^+ being on the physical sheet, and x^- on the other sheet.[13]

Once the differentials dp and dq have been defined by (4.3) and (4.4) with respect to a given equivalence class of cuts, one can move the cuts around into the other equivalence class (by crossing say $x = -1$ with a single cut) to obtain a representation of dp and dq with respect to the other equivalence class of cuts. So for instance, if we define dp and dq by (4.3) and (4.4) with respect to the equivalence class of cuts in figure 5 (a), then with respect to the equivalence class of cuts in figure 5 (b) the definition of dp will now be (4.4) and that of dq will now be (4.3).

In summary, both equivalence classes of cuts represents the very same algebraic curve Σ, but each equivalence class gives rise to a different definition of dp and dq. So the two equivalence classes of cuts give rise to two separate finite-gap solutions but which can be related by a $\tau \leftrightarrow \sigma$ transformation (4.2). Indeed, if in the construction of [34] we assume the generic configuration of cuts given in figure 5 (b), instead of figure 5 (a) as was assumed in [34], then the resulting solution is the generic helical string but with

$$X \leftrightarrow T$$

namely the 2D transformed helical string (3.6), (3.7). Therefore, with dp and dq defined as above by their respective asymptotics (4.3) and (4.4) at $x = \pm 1$, the helical string

[13]They should not be confused with AdS/CFT spectral parameters (5.3).

of [28, 34] is the general finite-gap solution corresponding to the class represented by figure 5 (a), whereas the 2D transformed helical string corresponds to the most general elliptic finite-gap solution on $\mathbb{R} \times S^3$ with cuts in the other class represented in figure 5 (b).

As is clear from the above, a given finite-gap solution is not associated with a particular equivalence class of cuts; since dp and dq are defined relative to an equivalence class of cuts, one can freely change equivalence class provided one also changes the definitions of dp and dq with respect to this new equivalence class according to (4.2), so that in the end dp and dq define the same differentials on Σ in either representation. For example, we can describe the 2D transformed helical string in two different ways: either we take the configuration of cuts in figure 5 (b) with dp and dq defined as usual by their asymptotics (4.3) and (4.4) at $x = \pm 1$, or we take the configuration of cuts in figure 5 (a) but need to swap the definitions of dp and dq in (4.3) and (4.4). In the following we will use the latter description of figure 5 (a) in order to take the singular limit $k \to 1$ where the cuts merge into a pair of singular points.

We can obtain expressions for the global charges $J_1 = (J_L + J_R)/2$, $J_2 = (J_L - J_R)/2$ along the same lines as in [34] for the helical string. In terms of the differential form

$$\alpha \equiv \frac{\sqrt{\lambda}}{4\pi}\left(x + \frac{1}{x}\right) dp\,, \qquad \tilde{\alpha} \equiv \frac{\sqrt{\lambda}}{4\pi}\left(x - \frac{1}{x}\right) dp\,, \tag{4.5}$$

we can write

$$J_1 = -\mathrm{Res}_{0^+}\alpha + \mathrm{Res}_{\infty^+}\alpha = \mathrm{Res}_{0^+}\tilde{\alpha} + \mathrm{Res}_{\infty^+}\tilde{\alpha}\,, \tag{4.6}$$

$$J_2 = -\mathrm{Res}_{0^+}\alpha - \mathrm{Res}_{\infty^+}\alpha\,. \tag{4.7}$$

Note that α and $\tilde{\alpha}$ both have simple poles at $x = 0\,, \infty$ but $\tilde{\alpha}$ also has simple poles at $x = \pm 1$ coming from the double poles in dp at $x = \pm 1$. It follows that we can rewrite (4.6), (4.7) as

$$J_1 = -\sum_{I=1}^{2} \frac{1}{2\pi i} \int_{\mathcal{A}_I} \tilde{\alpha} - \mathrm{Res}_{(+1)^+}\tilde{\alpha} - \mathrm{Res}_{(-1)^+}\tilde{\alpha}\,, \tag{4.8}$$

$$J_2 = \sum_{I=1}^{2} \frac{1}{2\pi i} \int_{\mathcal{A}_I} \alpha\,, \tag{4.9}$$

where $\mathcal{A}_I$ is the $\mathcal{A}$-cycle around the I-th cut. Whereas in [34] the residues of $\tilde{\alpha}$ at $x = \pm 1$ were of the same sign (as a consequence of $p(x)$ having equal residues at $x = \pm 1$) so that their sum gave the energy E of the string, in the present 2D-transformed helical case the residues of $\tilde{\alpha}$ at $x = \pm 1$ are now opposite (since $p(x)$ now has opposite residues at $x = \pm 1$) and therefore cancel in the above expression for J_1, resulting in the following expressions

$$-J_1 = \sum_{I=1}^{2} \frac{1}{2\pi i} \int_{\mathcal{A}_I} \tilde{\alpha}\,, \qquad J_2 = \sum_{I=1}^{2} \frac{1}{2\pi i} \int_{\mathcal{A}_I} \alpha\,. \tag{4.10}$$

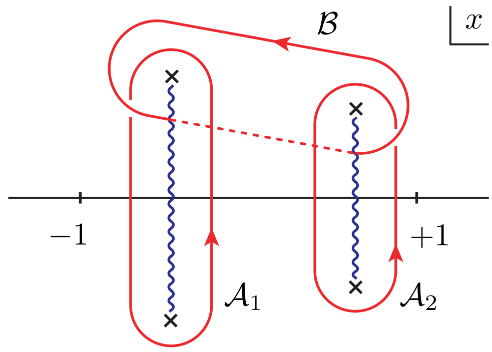

Figure 6: Definitions of cycles.

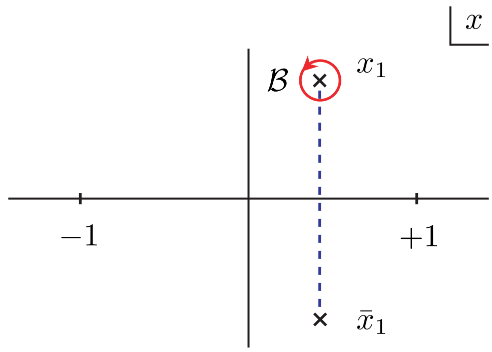

Figure 7: $k \to 1$ limit of cuts.

In parallel to the discussion of the helical string case in [34], there are two types of limits one can consider: the symmetric cut limit (where the curve acquires the extra symmetry $x \leftrightarrow -x$) which corresponds to taking $\omega_{1,2} \to 0$ in the finite-gap solution, or the singular curve limit which corresponds to taking the moduli of the curve to one, $k \to 1$. In the symmetric cut limit the discussion is identical to that in [34] (when working with the configuration of cuts in figure 5 (a)), in particular there are two possibilities corresponding to the type $(i)'$ and type $(ii)'$ cases, for which the cuts are symmetric with $x_1 = -\bar{x}_2$ and imaginary with $x_1 = -\bar{x}_1$, $x_2 = -\bar{x}_2$ respectively (see figure 2 of [34]).

In the singular limit $k \to 1$ where both cuts merge into a pair of singular points at $x = x_1$, $\bar{x}_1$ [34], the sum of $\mathcal{A}$-cycles turns into a sum of cycles around the points $x_1, \bar{x}_1$, so that (4.10) yields in this limit

$$-J_1 = \mathrm{Res}_{x_1}\tilde{\alpha} + \overline{\mathrm{Res}_{x_1}\tilde{\alpha}}\,, \qquad J_2 = \mathrm{Res}_{x_1}\alpha + \overline{\mathrm{Res}_{x_1}\alpha}\,. \tag{4.11}$$

Moreover, in the singular limit dp acquires simple poles at $x = x_1, \bar{x}_1$ so that the periodicity condition about the $\mathcal{B}$-cycle, $\int_{\mathcal{B}} dp = 2\pi n$, implies

$$\mathrm{Res}_{x_1} dp = \frac{n}{i}\,.$$

Let us set $n = 1$ (n can be easily recovered at any moment). Then (4.11) simplifies to

$$-J_1 = \frac{\sqrt{\lambda}}{4\pi} \left| \left(x_1 - \frac{1}{x_1} \right) - \left(\bar{x}_1 - \frac{1}{\bar{x}_1} \right) \right| , \tag{4.12}$$

$$J_2 = \frac{\sqrt{\lambda}}{4\pi} \left| \left(x_1 + \frac{1}{x_1} \right) - \left(\bar{x}_1 + \frac{1}{\bar{x}_1} \right) \right| . \tag{4.13}$$

The energy $E = \sqrt{\lambda}\,\kappa = (n\sqrt{\lambda}/\pi)\,\mathcal{E}$ diverges in the singular limit $k \to 1$, but this divergence can be related to the one in $\Delta\varphi_1$. In the present case the σ-periodicity condition $\int_{\mathcal{B}} dp \in 2\pi\mathbb{Z}$ can be written as (c.f., equation (2.23) in [34])

$$-\frac{2\mathbf{K}\sqrt{1-v^2}}{v} = \frac{2\pi}{n}\kappa' \equiv \frac{2\pi\kappa|x_1 - \bar{x}_2|}{n\sqrt{y_+ y_-}} ,$$

where $\mathbf{K} = \mathbf{K}(k)$, $y_\pm = y(x)|_{x=\pm 1} > 0$, $y(x) = (x - x_1)(x - \bar{x}_1)(x - x_2)(x - \bar{x}_2)$ and v can be expressed in the present setup as $v = \frac{y_+ - y_-}{y_+ + y_-}$ (see [34]). Using this σ-periodicity condition the energy can be expressed in the $k \to 1$ limit as

$$\mathcal{E} = \frac{u_1}{v}(1 - v^2)\,\mathbf{K}(1) .$$

We can relate this divergent expression with the expression (3.14) for $\Delta\varphi_1$ which also diverge in the limit $k \to 1$, making use of the relation $u_1 v = \tan\omega_1$ (see [34] where the notation is $u_1 = v_-$ and $\omega_1 = \tilde{\rho}_-$), and find

$$\mathcal{E} - \frac{\Delta\varphi_1}{2} = -\left(\omega_1 - \frac{(2n_1' + 1)\pi}{2} \right) \equiv \bar{\theta} . \tag{4.14}$$

Comparing this scenario with the one for helical strings in [34] we can write an expression for $\bar{\theta}$ in terms of the spectral data x_1 of the singular curve. Identifying

$$\bar{\theta} = -\frac{i}{2} \ln \left(\frac{x_1}{\bar{x}_1} \right) , \tag{4.15}$$

the expressions (4.12), (4.13) and (4.15) together imply the relation[14]

$$-J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2\bar{\theta}} . \tag{4.16}$$

5. Gauge theory duals

In view of the pulsating (oscillating) nature of the $\tau \leftrightarrow \sigma$ transformed helical strings we saw in the previous sections, the gauge theory operators dual to those classical strings should be made up not only of holomorphic but also of non-holomorphic scalars. In this section we discuss the gauge theory interpretation of 2D transformed strings, which includes a single-spike string and a static circular string.

[14] The sign difference between (3.27) and here is not essential.

First let us review some relevant aspects of the SU(2) magnon boundstates. Let Z or W be two of the three complex scalar fields of $\mathcal{N}=4$ SYM (the third one will be denoted Y). Then operators in the SU(2) sector take the forms $\mathcal{O} = \mathrm{Tr}\,(\Phi_{i_1}\Phi_{i_2}\ldots) + \ldots$ with each Φ_{i_l} ($l = 1, \ldots, L$) being either Z or W. The BPS operator $\mathrm{Tr}\,(ZZ\ldots)$ made up only of Z is the ferromagnetic ground state for the SYM spin-chain. In [16], it is shown that dyonic giant magnons are dual to magnon boundstates $\mathcal{O}_{\rm DGM} \sim \mathrm{Tr}\left(Z^K W^M\right) + \ldots$ in the SYM spin-chain ($K \to \infty$, M: finite), whose dispersion relation is given by

$$\Delta_{\mathcal{O}_{\rm DGM}} - K = \sqrt{M^2 + 16 g^2 \sin^2\left(\frac{P}{2}\right)}\,, \qquad g \equiv \frac{\sqrt{\lambda}}{4\pi}\,. \tag{5.1}$$

This agrees with the energy-spin relation for a dyonic giant magnon under the identifications $J_1 = K$ ($\to \infty$) and $J_2 = M$. Here $P = \sum_{j=1}^{M} p_j$ is the sum of the momenta p_j ($j = 1, \ldots, M$) of the constituent magnons. They satisfy the following boundstate condition,

$$x^-(p_j) = x^+(p_{j+1}) \qquad \text{for} \quad j = 1, \ldots, M\,, \tag{5.2}$$

where $x^\pm(p)$ are the standard AdS/CFT spectral parameters, defined by

$$x^\pm(u) = x\left(u \pm \frac{i}{2g}\right) \quad \text{where} \quad x(u) = \frac{1}{2}\left(u + \sqrt{u^2 - 4}\right)\,, \tag{5.3}$$

and $u = u(p)$ is the rapidity variable,

$$u(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1 + 16 g^2 \sin^2\left(\frac{p}{2}\right)}\,. \tag{5.4}$$

Now let us turn to the present oscillating case. First we discuss the two-spin single-spike string case. As we have seen, in contrast to the dyonic giant magnon, it has finite spins J_i ($i = 1, 2$) and infinite energy. This fact allows us to claim that the relevant dual SYM operators should look like

$$\mathcal{O}_{\rm SS} = \mathrm{Tr}\left(Z^K\, \overline{Z}^{K'}\, W^M\, \mathcal{S}^{(L-K-K'-M)/2}\right) + \ldots\,, \quad L\,, K\,, K' \to \infty\,, \quad K - K'\,, M : \text{finite}\,. \tag{5.5}$$

In (5.5), the factor $\mathcal{S}$ appearing in (5.5) is the SO(6)-singlet composite[15]

$$\mathcal{S} \sim Z\overline{Z} + W\overline{W} + Y\overline{Y}\,. \tag{5.6}$$

One can easily understand that the pairs like $Z\overline{Z}$ give rise to oscillating motion in the sting side, since if we associate Z to a particle rotating along a great circle of S^5 clockwise, the other particle associated with $\overline{Z}$ rotates counterclockwise, thus making the string connecting these two points non-rigid and oscillating. The dots in (5.5) denotes terms that mix

[15]The SO(6) sector is not closed beyond one-loop level in λ, and operator mixing occurs in the full $PSU(2,2|4)$ sector due to the higher-loop effects. So one might think $\mathcal{S}$ should be a $PSU(2,2|4)$ singlet rather than an SO(6) singlet. However, we can still expect that such mixing into $PSU(2,2|4)$ is suppressed in our classical ($L \to \infty$) setup as in [47]. We would like to thank J. Minahan for discussing this point.

JHEP11(2007)033

under renormalization. An important assumption is that M Ws form a boundstate. Indeed loop-effects mix $Z\overline{Z}$ with other neutral combinations $W\overline{W}$ and $Y\overline{Y}$, but it is assumed the boundstate condition still holds. Let $X^{\pm}$ be the spectral parameters assigned to the boundstate. We write them as

$$X^{\pm} = R\, e^{\pm iP/2} \quad \text{with} \quad R = \frac{M + \sqrt{M^2 + 16g^2 \sin^2(P/2)}}{4g \sin(P/2)} \; (>1)\,, \tag{5.7}$$

where P is the momentum carried by the boundstate. Recall that we took $\mathrm{Tr}\,(ZZ\ldots)$ as the vacuum state, therefore W is an excitation above the vacuum with $\Delta_0 - J_1 = 1$,[16] whereas $\overline{Z}$ is an excitation with $\Delta_0 - J_1 = 2$.[17] The composite $\mathcal{S}$ also contributes to the spin-chain energy in some way, and we must take all the contributions into account when evaluating the total energy $\Delta_{\mathcal{O}_{\rm SS}} - J_1$ of (5.5). We assume that the contribution of M Ws results in two parts; one is the boundstate energy that contributes in the same way as in the case of an SU(2) boundstate $\mathcal{O}_{\rm DGM} \sim \mathrm{Tr}\left(Z^K W^M\right) + \ldots\ (K \to \infty)$, and the other is its interactions with other fields. One can then write down the total energy as

$$\Delta_{\mathcal{O}_{\rm SS}} - (K - K') = \frac{g}{i}\left[\left(X^+ - \frac{1}{X^+}\right) - \left(X^- - \frac{1}{X^-}\right)\right] + \chi\,. \tag{5.8}$$

The first term in r.h.s. comes from the boundstate W^M, while the last χ accounts for contributions concerning $\mathcal{S}$, $\overline{Z}$ and all their interactions with other fields, including Ws. Currently we have no knowledge of how the actual form of χ looks like, and so we leave it as some function of the coupling and boundstate momentum here (however, we will later discuss its form in the strong coupling, infinite-winding limit). One can also express the J_2-charge carried by the boundstate in terms of the spectral parameters as

$$M = \frac{g}{i}\left[\left(X^+ + \frac{1}{X^+}\right) - \left(X^- + \frac{1}{X^-}\right)\right]. \tag{5.9}$$

Now perform a change of basis for the spin-chain, and take $\mathrm{Tr}\left(\overline{Z}\,\overline{Z}\ldots\right)$ as the vacuum state, instead of $\mathrm{Tr}\,(ZZ\ldots)$. This particular transformation of susy multiplet, namely the charge conjugation, maps the original W^M to $\overline{W}^M$ with new spectral parameters

$$\widetilde{X}^{\pm} = 1/X^{\pm}\,. \tag{5.10}$$

This is actually a crossing transformation that maps a usual particle to its conjugate particle (antiparticle) [7]. In the new basis, $\overline{W}$s, Zs and $\overline{\mathcal{S}} = \mathcal{S}$ play the role of excitations above the new vacuum. The contribution of $\mathcal{S}$ to the new vacuum should be the same as in the old case since it is an SO(6) singlet, and we assume the total contributions from all excitations to be the same as in the old case. Then one obtains a relation similar to (5.8),

$$\Delta_{\mathcal{O}_{\rm SS}} - (K' - K) = \frac{g}{i}\left[\left(\widetilde{X}^+ - \frac{1}{\widetilde{X}^+}\right) - \left(\widetilde{X}^- - \frac{1}{\widetilde{X}^-}\right)\right] + \chi\,, \tag{5.11}$$

[16] We follow a convention such that a Z field has $\Delta_0 - J_1 = 0$, where Δ_0 denotes the bare dimension.

[17] In fact, $\overline{Z}$ is not a fundamental excitation. We should regard it as an excitation corresponding to a two-magnon state.

and similarly for the second charge. From (5.8)–(5.11), it follows that

$$\Delta_{\mathcal{O}_{\rm SS}} = \chi \qquad \text{and} \qquad K' - K = \sqrt{M^2 + 16g^2 \sin^2\left(\frac{P}{2}\right)} \,. \tag{5.12}$$

Then if we identify naturally

$$K - K' \equiv J_1 \,, \quad M \equiv J_2 \quad \text{and} \quad P \equiv 2\pi m \pm 2\bar{\theta} \quad (m \in \mathbb{Z}\,;\ 0 \le \bar{\theta} \le \pi/2) \,, \tag{5.13}$$

the second relation in (5.12) precisely reproduces the dispersion relation for single-spike strings, after substituting $g^2 = \lambda/16\pi^2$. Here we included an integer degree of freedom m that plays the role of the winding number in the string theory side. One can also deduce that

$$\frac{J_2}{J_1} = \frac{R^2 - 1}{R^2 + 1} \,, \tag{5.14}$$

which corresponds to $\sin\gamma$ in the notation used in [36]. In (5.13), one may choose either the plus/minus signs in P ; they correspond to the momenta of a particle/antiparticle.

Notice also the above argument, resulting in

$$-J_1 = \frac{g}{i}\left[\left(X^+ - \frac{1}{X^+}\right) - \left(X^- - \frac{1}{X^-}\right)\right] , \tag{5.15}$$

$$J_2 = \frac{g}{i}\left[\left(X^+ + \frac{1}{X^+}\right) - \left(X^- + \frac{1}{X^-}\right)\right] , \tag{5.16}$$

is consistent with what we found in the previous section, (4.12) and (4.13), if we, as usual, identify the string theory spectral parameters x_1 and $\bar{x}_1$ (in finite-gap language) with the ones for gauge theory X^+ and X^- (for the boundstate).

To proceed in the reasoning, suppose the asymptotic behavior of χ in the strong coupling and infinite-"winding" limit becomes

$$\chi \sim 2gP = m\sqrt{\lambda} \pm \frac{\bar{\theta}}{\pi} \,, \qquad (m \to \infty) \,. \tag{5.17}$$

We kept here $\pm\bar{\theta}/\pi$ term to ensure that χ is not just given by (integer) $\times \sqrt{\lambda}$ but contains some continuous shift away from that. We will give more explanations concerning this conjecture soon. The relation (5.17) then implies that

$$\Delta_{\mathcal{O}_{\rm SS}} - \frac{\sqrt{\lambda}}{2\pi} \cdot 2\pi m = \pm \frac{\sqrt{\lambda}}{\pi}\, \bar{\theta} \,, \tag{5.18}$$

where we used the identifications we made before. This can be compared to the string theory result for the single-spike, (3.28). The integer m here corresponds to the winding number N_1 there (recall that for single spike case, we had $\Delta\varphi_1 = 2\pi N_1$ due to the periodicity condition). When there are n boundstates in the spin-chain all with the same momentum P , r.h.s. of (5.18) is just multiplied by n and modified to $n(\sqrt{\lambda}/\pi)\,\bar{\theta}$, which corresponds to an array of n single-spikes.

Let us explain the conjecture (5.17) in greater detail. Of course one of the motivations is that it reproduces the relation (5.18) of the string side, as we have just seen. Further evidence can be found by considering particular sets of operators contained in (5.5) and checking for consistency. For example, let us consider the limit $K - K' \to 0$ and $M \to 0$. This takes the operator (5.5) to the form $\mathrm{Tr}\left((Z\overline{Z})^K \mathcal{S}^{L/2-K}\right) + \dots$, which must sum up to the singlet operator $\mathrm{Tr}\,\mathcal{S}^{L/2}$ for it to be a solution of the Bethe ansatz equation. In this limit, the "angle" $\bar{\theta}$ should vanish in view of the second equation in (5.12) and (5.13). Therefore the relation (5.17) together with the first equation in (5.12) imply that the canonical dimension of the singlet operator is just given by

$$\Delta_{\mathrm{Tr}\,\mathcal{S}^{L/2}}\big|_{L\to\infty} = m\sqrt{\lambda}\,, \qquad (m \to \infty)\,, \tag{5.19}$$

which agrees with the energy expression (3.32) of the $\tau \leftrightarrow \sigma$ transformed point-like BPS string (in the limit $\mu\sqrt{U} \to \infty$), under the identification $N_2 = m$.

As we have seen, in contrast to the dyonic giant magnon vs. magnon bound state $\mathcal{O}_{\mathrm{DGM}} \sim \mathrm{Tr}\left(Z^\infty W^M\right) + \dots$ case, the correspondence between two-spin single-spike vs. $\mathcal{O}_{\mathrm{SS}}$ given in (5.5) is slightly more involved. In the former correspondence in the infinite spin sector, the magnon boundstate is an excitation above the BPS vacuum $\mathcal{O}_{\mathrm{F}} \sim \mathrm{Tr}\,(Z^\infty)$, and one can think of the boundstate W^M as the counterpart of the corresponding dyonic giant magnon. For the latter case in the infinite winding sector, however, it is not the boundstate W^M alone but the "$Z^K\,\overline{Z}^{K'}\,W^M + \dots$" part of $\mathcal{O}_{\mathrm{SS}}$ that encodes the single-spike. It can be viewed as an excitation above the SO(6) singlet operator $\mathcal{O}_{\mathrm{AF}} \sim \mathrm{Tr}\,\mathcal{S}^{L/2}$. Actually this is the "antiferromagnetic" state of the SO(6) spin-chain, which is "as far from BPS as possible" (Notice that a solution of the Bethe ansatz equation with $J_1 = J_2 = J_3 = 0$ is nothing but the SO(6) singlet state). It is dual to the rational circular static string (3.30) obtained by performing a $\tau \leftrightarrow \sigma$ transformation on the point-like BPS string.

6. Summary and discussions

In the previous works [28, 34], three of the current authors constructed the most general elliptic ("two-cut") classical string solutions on $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$, called helical strings. They were shown to include various strings studied in the *large-spin* sector. Schematically, the family tree reads

$$\mathrm{I}: \quad \begin{array}{l}\text{Type } (i) \text{ helical string}\\ \text{with generic } k \text{ and } \omega_{1,2}\end{array} \longrightarrow \begin{cases} \text{- Point-like (BPS), rotating string } (k \to 0) \\ \text{- Array of dyonic giant magnons } \quad (k \to 1) \\ \text{- Elliptic, spinning folded string } \quad (\omega_{1,2} \to 0) \end{cases} ,$$

$$\mathrm{II}: \quad \begin{array}{l}\text{Type } (ii) \text{ helical string}\\ \text{with generic } k \text{ and } \omega_{1,2}\end{array} \longrightarrow \begin{cases} \text{- Rational, spinning circular string } (k \to 0) \\ \text{- Array of dyonic giant magnons } \quad (k \to 1) \\ \text{- Elliptic, spinning circular string } \quad (\omega_{1,2} \to 0) \end{cases} .$$

Moreover, the single-spin limit of the type (i) helical strings agrees with so-called "spiky strings" studied in [23, 20].[18]

[18] The two-spin helical strings are different from the spiky strings in that they have no singular points in spacetime. When embedded in $\mathbb{R} \times S^3$, the singular "cusps" of the spiky string that apparently existed on $\mathbb{R} \times S^2$ are all smoothed out to result in non-spiky profiles.

For Cases I and II, the gauge theory duals are also well-known. They are all of the form

$$\mathcal{O} \sim \mathrm{Tr}\left(Z^{L-M} W^{M}\right) + \dots\,, \tag{6.1}$$

with L very large. For example, for the type (i) case, a BPS string ($k \to 0$) of course corresponds to $M = 0$, and a BMN string corresponds to M very small. A dyonic giant magnon corresponds to an M-magnon boundstate in the asymptotic SYM spin-chain ($L \to \infty$), which is described by a straight Bethe string in rapidity plane [17, 19]. In the Bethe string, all M roots are equally spaced in the imaginary direction, reflecting the pole condition of the asymptotic S-matrix. As to the elliptic folded/circular strings, they correspond to, respectively, the so-called double-contour/imaginary-root distributions of Bethe roots [48].

In contrast, in the current paper, we explored non-holomorphic sector of classical strings on $\mathbb{R} \times S^3$, and found a new interpolation. This includes a *large-winding* sector where $m\sqrt{\lambda}$ becomes of the same order as the energy which diverges (m being the winding number). We saw that when classical strings on $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$ are considered in conformal gauge, an operation of interchanging τ and σ, as well as keeping temporal gauge $t \propto \tau$, maps the original helical strings to another type of helical strings. Roughly speaking, rotating/spinning solutions with large spins became oscillating solution with large windings. Again, schematically, we found:

$$\mathrm{I}' : \quad \begin{array}{l}\text{Type } (i)' \text{ helical string}\\ \text{with generic } k \text{ and } \omega_{1,2}\end{array} \quad \longrightarrow \quad \begin{cases} \text{- Rational, static circular string} & (k \to 0) \\ \text{- Array of single-spike strings} & (k \to 1) \\ \text{- Elliptic, type } (i)' \text{ pulsating string} & (\omega_{1,2} \to 0) \end{cases} \,,$$

$$\mathrm{II}' : \quad \begin{array}{l}\text{Type } (ii)' \text{ helical string}\\ \text{with generic } k \text{ and } \omega_{1,2}\end{array} \quad \longrightarrow \quad \begin{cases} \text{- Rational circular string} & (k \to 0) \\ \text{- Array of single-spike strings} & (k \to 1) \\ \text{- Elliptic, type } (ii)' \text{ pulsating string} & (\omega_{1,2} \to 0) \end{cases} \,.$$

In section 4, we investigated 2D-transformed helical strings from the finite-gap perspective. We were able to understand the effect of the $\tau \leftrightarrow \sigma$ operation as an interchange of quasi-momentum and quasi-energy. The transformed helical strings were described as general two-cut finite-gap solutions as in the original case [34], the only difference being the asymptotic behaviors of differentials at $x \to \pm 1$ (or equivalently, different configurations of cuts with respect to interval $(-1, 1)$). By expressing the charges in terms of spectral parameters (branch-points of the cuts), the charge relations for single spikes were also reproduced.

In section 5, the gauge theory duals of the $\tau \leftrightarrow \sigma$ transformed strings (derivatives of type $(i)'$ and $(ii)'$ helical strings) were identified with operators of the form

$$\mathcal{O} \sim \mathrm{Tr}\left(Z^{K}\, \overline{Z}^{K'}\, W^{M}\, \mathcal{S}^{(L-K-K'-M)/2}\right) + \dots\,, \tag{6.2}$$

with $\mathcal{S}$ the SO(6) singlet composite (5.6). The single-spike limit $k \to 1$ was identified with the $K\,, K' \to \infty$ limit while keeping $K - K'$ and M finite (see (5.5)). In this limit, the "$Z^{K}\, \overline{Z}^{K'}\, W^{M} + \dots$" part in the operator, of which W^{M} is assumed to form a boundstate, was claimed to be responsible for the transverse excitation (spikes) of the string state winding infinitely many times around a great circle of S^5. In other words, the spikes are dual to excitations above the "antiferromagnetic" state $\mathrm{Tr}\,\mathcal{S}^{L/2}$. This "antiferromagnetic" state is the singlet state of the SO(6) spin-chain, and located "as far from BPS as possible" in the spin-chain spectrum. These features can be compared to that of magnons in the large spin sector (impurity above BPS vacuum) corresponding to the transverse excitations of the point-like string orbiting around a great circle of S^5.

It would be interesting to check the prediction (5.17) directly by using the conjectured AdS/CFT Bethe ansatz equation. In the SU(2) sector where the number of operators is finite, the nature of the antiferromagnetic state is better understood [49], and the upper bound on the energy is known [50] (see also [51]). It is proportional to $\sqrt{\lambda}$, which is the same behavior as our conjecture (5.17). Recall that we argued the SO(6) singlet state was dual to a large winding string state with zero-spins, (3.30). If the prediction (5.17) is correct, then we should be able to reproduce it by the SO(6) Bethe ansatz equation approach. An approach similar to [50] would be useful. In this case, the "spiky magnon" part "$Z^K\,\overline{Z}^{K'}\,W^M + \ldots$" could be understood as (macroscopic number of) "holes" made in the continuous mode numbers associated with the SO(6) singlet Bethe root configuration.[19] The SO(6) singlet state was also studied in [52], where an integral equation for the Bethe root density was derived. It would be interesting to study it at strong coupling and compare it with our results.[20]

Since the $\tau \leftrightarrow \sigma$ transformed string solutions discussed in this paper are periodic classical solutions, one can define corresponding action variables, namely the oscillation numbers. By imposing the Bohr-Sommerfeld quantization condition, one obtains integer valued action variables, which from lesson of the large spin sector [16] we can again expect to correspond to filling fractions defined for the SO(6) spin-chain. It would be interesting to understand this correspondence from the finite-gap perspective along the lines of [35, 53].

It would be also interesting to compare the spectra of AdS/CFT near the SO(6) "antiferromagnetic" vacuum by an effective sigma model approach (without any apparent use of integrability) [54]. In the SU(2) case, a similar approach was taken in [51], where a continuum limit of the half-filled Hubbard chain was compared to an effective action for "slow-moving" strings with $J_1 = J_2$. In our case, some Hubbard-like model with SO(6) symmetry would give clues.

We hope to revisit these issues in other publications in the near future.

Note added After the submission of the first version of our paper to arXiv.org `0709.4033 [hep-th]` for publication, we learned that the paper `0709.4231 [hep-th]` [55] appeared, in which single-spike strings are generalized to three-spin cases. We thank N. P. Bobev and R. C. Rashkov for correspondence.

[19] In the weak coupling regime, the SO(6) singlet Bethe root configuration and excitations above it were studied in [2, 45, 47].

[20] We thank M. Staudacher for pointing this out to us.

Acknowledgments

We acknowledge useful discussions with N. Dorey, Y. Hatsuda, Y. Imamura, J. Minahan, M. Staudacher, A.A. Tseytlin and K. Zarembo. We thank Y. Imamura, J. Minahan and A.A. Tseytlin for reading the draft carefully and giving us illuminating comments. KO is grateful to University of Cambridge, Centre for Mathematical Sciences, for its warm hospitality during the work was done. RS thanks the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the YITP workshop YITP-W-07-05 on "String Theory and Quantum Field Theory" were useful to complete this work. The work of KO is supported in part by JSPS Research Fellowships for Young Scientists. The work of BV was supported by EPSRC.

A. Helical strings on $AdS_3 \times S^1$

This appendix is devoted to helical string solutions in the SL(2) sector. The construction almost parallels that in [28], however, non-compactness of the AdS space lead to new non-trivial features compared to the sphere case.

A.1 Classical strings on $AdS_3 \times S^1$ and complex sinh-Gordon model

A string theory on $AdS_3 \times S^1 \subset AdS_5 \times S^5$ spacetime is described by an $O(2,2) \times O(2)$ sigma model. Let us denote the coordinates of the embedding space as η_0 , η_1 (for AdS_3) and ξ_1 (for S^1) and set the radii of AdS_3 and S^1 both to unity,

$$\vec{\eta}^{\,*} \cdot \vec{\eta} \equiv -\left|\eta_0\right|^2 + \left|\eta_1\right|^2 = -1\,, \qquad \left|\xi_1\right|^2 = 1\,. \tag{A.1}$$

In the standard polar coordinates, the embedding coordinates are expressed as

$$\eta_0 = \cosh\rho\, e^{it}\,, \quad \eta_1 = \sinh\rho\, e^{i\phi_1}\,, \quad \xi_1 = e^{i\varphi_1}\,, \tag{A.2}$$

and all the charges of the string states are defined as Nöther charges associated with shifts of the angular variables. The bosonic Polyakov action for the string on $AdS_3 \times S^1$ is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\gamma^{ab}\left(\partial_a \vec{\eta}^{\,*} \cdot \partial_b \vec{\eta} + \partial_a \xi^* \cdot \partial_b \xi\,\right) + \widetilde{\Lambda}\Big(\vec{\eta}^{\,*} \cdot \vec{\eta} + 1\Big) + \Lambda\Big(\xi_1^* \cdot \xi_1 - 1\Big)\right]\,, \tag{A.3}$$

and we take the same conformal gauge as in the $\mathbb{R} \times S^3$ case. From the action (A.3) we get the equations of motion

$$\partial_a \partial^a \vec{\eta} - \left(\partial_a \vec{\eta}^{\,*} \cdot \partial^a \vec{\eta}\right) \vec{\eta} = 0\,, \qquad \partial_a \partial^a \xi_1 + \left(\partial_a \xi_1^* \cdot \partial^a \xi_1\right) \xi_1 = 0\,, \tag{A.4}$$

and Virasoro constraints

$$0 = \mathcal{T}_{\sigma\sigma} = \mathcal{T}_{\tau\tau} = \frac{\delta^{ab}}{2}\left(\partial_a \vec{\eta}^{\,*} \cdot \partial_b \vec{\eta} + \partial_a \xi_1^* \cdot \partial_b \xi_1\right)\,, \tag{A.5}$$

$$0 = \mathcal{T}_{\tau\sigma} = \mathcal{T}_{\sigma\tau} = \mathrm{Re}\,\left(\partial_\tau \vec{\eta}^{\,*} \cdot \partial_\sigma \vec{\eta} + \partial_\tau \xi_1 \cdot \partial_\sigma \xi_1^*\right)\,. \tag{A.6}$$

The PLR reduction procedure, which we made use of in obtaining the $O(4)$ sigma model solutions from Complex sine-Gordon solution, also works for the current case in much the same way. The $O(2,2)$ sigma model in conformal gauge is now related to what we call Complex sinh-Gordon (CshG) model, which is defined by the Lagrangian

$$\mathcal{L}_{\rm CshG} = \frac{\partial^a \psi^* \partial_a \psi}{1+\psi^*\psi} + \psi^*\psi\,, \tag{A.7}$$

with $\psi = \psi(\tau,\sigma)$ being a complex field. It can be viewed as a natural generalization of the well-known sinh-Gordon model in the sense we describe below. By defining two real fields α and β of the CshG model through $\psi \equiv \sinh(\alpha/2)\exp(i\beta/2)$, the Lagrangian (A.7) is rewritten as

$$\mathcal{L}_{\rm CshG} = \frac{1}{4}\,(\partial_a\alpha)^2 + \frac{\tanh^2(\alpha/2)}{4}\,(\partial_a\beta)^2 + \sinh^2(\alpha/2)\,. \tag{A.8}$$

The equations of motion that follow from the Lagrangian are

$$\partial^a\partial_a\psi - \psi^*\frac{\partial^a\psi\,\partial_a\psi}{1+\psi^*\psi} - \psi\,(1+\psi^*\psi) = 0\,, \tag{A.9}$$

$$\text{i.e.,}\quad \begin{cases} \partial^a\partial_a\alpha - \dfrac{\sinh(\alpha/2)}{2\cosh^3(\alpha/2)}\,(\partial_a\beta)^2 - \sinh\alpha = 0\,, \\[2ex] \partial^a\partial_a\beta + \dfrac{2\,\partial_a\alpha\,\partial^a\beta}{\sinh\alpha} = 0\,. \end{cases} \tag{A.10}$$

We refer to the coupled equations (A.10) as Complex sinh-Gordon (CshG) equations. If β is a constant field, the first equation in (A.10) reduces to

$$\partial_a\partial^a\alpha - \sinh\alpha = 0\,, \tag{A.11}$$

which is the ordinary sinh-Gordon equation. As readers familiar with the PLR reduction can easily imagine, it is this field α that gets into a self-consistent potential in the Schrödinger equation this time. Namely, we can write the string equations of motion given in (A.4) as

$$\partial_a\partial^a\vec{\eta} - (\cosh\alpha)\,\vec{\eta} = 0\,, \qquad \cosh\alpha \equiv \partial_a\vec{\eta}^{\,*}\cdot\partial^a\vec{\eta}\,, \tag{A.12}$$

with the same field α we introduced as the real part of the CshG field ψ . What this means is that if $\{\vec{\eta},\xi\}$ is a consistent string solution which satisfies Virasoro conditions (A.5) and (A.6), then $\psi = \sinh(\alpha/2)\exp(i\beta/2)$ defined via (A.12) and (A.16) solves the CshG equations.

The derivation of this fact parallels the usual PLR reduction procedure. Let us define worldsheet light-cone coordinates as $\sigma^{\pm} = \tau \pm \sigma$, and the embedding coordinates as $\eta_0 = Y_0 + iY_5$ and $\eta_1 = Y_1 + iY_2$. Then consider the equations of motion of the $O(2,2)$ nonlinear sigma model through the constraints

$$\vec{Y}\cdot\vec{Y} = -1\,,\quad (\partial_+\vec{Y})^2 = -1\,,\quad (\partial_-\vec{Y})^2 = -1\,,\quad \partial_+\vec{Y}\cdot\partial_-\vec{Y} \equiv -\cosh\alpha\,, \tag{A.13}$$

where $\vec{Y}\cdot\vec{Y} \equiv (\vec{Y})^2 \equiv -(Y_0)^2 + (Y_1)^2 + (Y_2)^2 - (Y_5)^2$. A basis of $O(2,2)$-covariant vectors can be given by Y_i , ∂_+Y_i , ∂_-Y_i and $K_i \equiv \epsilon_{ijkl}Y^j\partial_+Y^k\partial_-Y^l$. By defining a pair of scalar functions u and v as

$$u \equiv \frac{\vec{K} \cdot \partial_+^2 \vec{Y}}{\sinh \alpha} \,, \qquad v \equiv \frac{\vec{K} \cdot \partial_-^2 \vec{Y}}{\sinh \alpha} \,, \tag{A.14}$$

the equations of motion of the $O(2,2)$ sigma model are recast in the form

$$\partial_- \partial_+ \alpha + \sinh \alpha + \frac{uv}{\sinh \alpha} = 0 \,, \qquad \partial_- u = \frac{v \, \partial_+ \alpha}{\sinh \alpha} \,, \qquad \partial_+ v = \frac{u \, \partial_- \alpha}{\sinh \alpha} \,. \tag{A.15}$$

One can easily confirm that this set of equations is equivalent to the pair of equations (A.10) of CshG theory, under the identifications

$$u = (\partial_+ \beta) \tanh \frac{\alpha}{2} \,, \qquad v = -(\partial_- \beta) \tanh \frac{\alpha}{2} \,. \tag{A.16}$$

Thus there is a (classical) equivalence between the $O(2,2)$ sigma model $\leftrightarrow$ CshG as in the $O(4) \leftrightarrow$ CsG case. Making use of the equivalence, one can construct classical string solutions on $AdS_3 \times S^1$ by the following recipe:

1. Find a solution ψ of CshG equation (A.9).
2. Identify $\cosh \alpha \equiv \partial_a \vec{\eta}^* \cdot \partial^a \vec{\eta}$, where α appears in the real part of the solution ψ , and η are the embedding coordinates of the corresponding string solution in AdS_3 .
3. Solve the "Schrödinger equation" (A.12) together with the Virasoro constraints (A.5) and (A.6), under appropriate boundary conditions.
4. Resulting set of $\vec{\eta}$ ("wavefunction") and ξ_1 gives the corresponding string profile in $AdS_3 \times S^1$.

Let us start with step 1. From the similarities between the CshG equation and the CsG equation, it is easy to find helical-wave solutions of the CshG equation. Here we give two such solutions that will be important later. The first one is given by

$$\psi_{\rm cd} = kc \frac{\mathrm{cn}(cx_v)}{\mathrm{dn}(cx_v)} \exp\left(i \sqrt{(1+c^2)(1+k^2c^2)} \, t_v \right) , \tag{A.17}$$

and the second one is

$$\psi_{\rm ds} = c \frac{\mathrm{dn}(cx_v)}{\mathrm{sn}(cx_v)} \exp\left(i \sqrt{(1-k^2c^2)(1+c^2-k^2c^2)} \, t_v \right) . \tag{A.18}$$

By substituting the solution (A.18) into the string equations of motion (A.12), we obtain

$$\left[-\partial_T^2 + \partial_X^2 - k^2 \left(\frac{2}{k^2 \, \mathrm{sn}^2(X,k)} - 1 \right) \right] \vec{\eta} = U \vec{\eta} \,, \tag{A.19}$$

under the identification of $(\mu\tau, \mu\sigma) \equiv (ct, cx)$. The "eigenenergy" U can be treated as a free parameter as was the case in [28]. Different choices of helical-waves of CshG equation simply correspond to taking different ranges of U .

We are now at the stage of constructing the corresponding string solution by following the steps 2 - 4 listed before. However, we do not need to do this literally. Since the metrics of $AdS_3 \times S^1$

$$ds^2_{AdS_3 \times S^1} = -\cosh^2 \tilde{\rho} \, d\tilde{t}^{\,2} + d\tilde{\rho}^2 + \sinh^2 \tilde{\rho} \, d\tilde{\phi}_1^2 + d\tilde{\varphi}_1^2 \,, \tag{A.20}$$

and of $\mathbb{R}\times S^3$

$$ds^2_{\mathbb{R}\times S^3} = -dt^2 + d\gamma^2 + \cos^2\gamma\, d\varphi_1^2 + \sin^2\gamma\, d\varphi_2^2\,, \tag{A.21}$$

are related by analytic continuation

$$\tilde\rho \leftrightarrow i\gamma, \quad \tilde t \leftrightarrow \varphi_1, \quad \tilde\phi_1 \leftrightarrow \varphi_2, \quad \tilde\varphi_1 \leftrightarrow t \quad \Longrightarrow \quad ds^2_{AdS_3\times S^1} \leftrightarrow -ds^2_{\mathbb{R}\times S^3}\,, \tag{A.22}$$

string solutions on both manifolds are related by a sort of analytic continuation of global coordinates. Therefore, the simplest way to obtain helical string solutions on $AdS_3 \times S^1$ is to perform analytic continuation of helical string solutions on $\mathbb{R}\times S^3$, as will be done in the following sections. Large parts of the calculation parallel the $\mathbb{R}\times S^3$ case. The most significant difference lies in the constraints imposed on the solution of the equations of motion, such as the periodicity conditions.

A.2 Helical strings on $AdS_3 \times S^1$ with two spins

In this section, we consider the analytic continuation of helical strings on $\mathbb{R}\times S^3$ to those on $AdS_3\times S^1$. Among various possible solutions, we will concentrate on two particular examples that have clear connections with known string solutions of interest to us. The first example, called type (iii) helical string, is a helical generalization of the folded string solution on $AdS_3\times S^1$ [56]. The second one, called type (iv), reproduces the SL(2) "giant magnon" solution [21, 29] in the infinite-spin limit.

A.2.1 Type (iii) helical strings

In [57], it was pointed out that (S,J) folded strings can be obtained from (J_1,J_2) folded strings by analytic continuation of the elliptic modulus squared, from $k^2\geq 0$ to $k^2 \leq 0$. Here we apply the same analytic continuation to type (i) helical strings to obtain solutions on $AdS_3\times S^1$, which we call type (iii) strings. For notational simplicity, it is useful to introduce a new moduli parameter q through the relation

$$k \equiv \frac{iq}{q'} \equiv \frac{iq}{\sqrt{1-q^2}}\,. \tag{A.23}$$

If k is located on the upper half of the imaginary axis, *i.e.*, $k = i\kappa$ with $0\leq\kappa$, then q is a real parameter in the interval $[0,1]$.

As shown in appendix B, the transformation (A.23) can be regarded as a T-transformation of the modulus τ. Hence, by performing a T-transformation on the profile of type (i) helical strings (3.1)-(3.3), we obtain type (iii) string solutions:

$$\eta_0 = \frac{C}{\sqrt{qq'}}\frac{\Theta_3(0)\,\Theta_0(\tilde X - i\tilde\omega_0)}{\Theta_2(i\tilde\omega_0)\,\Theta_3(\tilde X)}\,\exp\Big(Z_2(i\tilde\omega_0)\tilde X + i\tilde u_0 \tilde T\Big)\,, \tag{A.24}$$

$$\eta_1 = \frac{C}{\sqrt{qq'}}\frac{\Theta_3(0)\,\Theta_1(\tilde X - i\tilde\omega_1)}{\Theta_3(i\tilde\omega_1)\,\Theta_3(\tilde X)}\,\exp\Big(Z_3(i\tilde\omega_1)\tilde X + i\tilde u_1 \tilde T\Big)\,, \tag{A.25}$$

$$\xi_1 = \exp\Big(i\tilde a \tilde T + i\tilde b\tilde X\Big)\,, \tag{A.26}$$

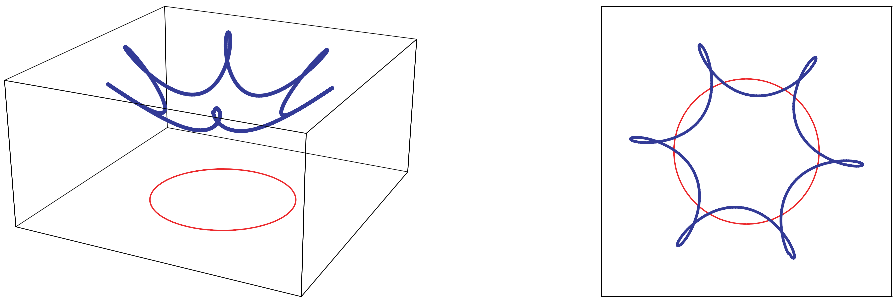

Figure 8: Type (iii) helical string ($q = 0.700$, $U = 12.0$, $\tilde{\omega}_0 = -0.505$, $\tilde{\omega}_1 = 0.776$, $n = 6$), projected onto AdS_2 spanned by $(\mathrm{Re}\,\eta_1, \mathrm{Im}\,\eta_1, |\eta_0|)$. The circle represents a unit circle $|\eta_1| = 1$ at $\eta_0 = 0$.

where we rescaled various parameters as

$$\tilde{X} = X/q', \quad \tilde{T} = T/q', \quad \tilde{\omega}_j = \omega_j/q', \quad \tilde{a} = a q', \quad \tilde{b} = b q', \quad \tilde{u}_j = u_j q'. \tag{A.27}$$

We choose the constant C so that they satisfy $|\eta_0|^2 - |\eta_1|^2 = 1$. One such possibility is to choose[21]

$$C = \left(\frac{1}{q^2 \,\mathrm{cn}^2(i\tilde{\omega}_0)} + \frac{\mathrm{sn}^2(i\tilde{\omega}_1)}{\mathrm{dn}^2(i\tilde{\omega}_1)} \right)^{-1/2}. \tag{A.28}$$

With the help of various formulae on elliptic functions, one can check that $\vec{\eta}$ in (A.24), (A.25) certainly solves the string equations of motion as

$$\left[-\partial_{\tilde{T}}^2 + \partial_{\tilde{X}}^2 + q^2 \left(2(1-q^2) \frac{\mathrm{sn}^2}{\mathrm{dn}^2}(\tilde{X}, q) - 1 \right) \right] \vec{\eta} = \tilde{U} \vec{\eta}, \tag{A.29}$$

if the parameters are related as

$$\tilde{u}_0^2 = \tilde{U} - (1-q^2) \frac{\mathrm{sn}^2(i\tilde{\omega}_0)}{\mathrm{cn}^2(i\tilde{\omega}_0)}, \qquad \tilde{u}_1^2 = \tilde{U} + \frac{1-q^2}{\mathrm{dn}^2(i\tilde{\omega}_1)}. \tag{A.30}$$

As is clear from (A.29), the type (iii) solution is related to the helical-wave solution of the CshG equation given in (A.17). The Virasoro constraints (A.5) and (A.6) impose constraints on $\tilde{a}$ and $\tilde{b}$ in (A.26):[22]

$$\tilde{a}^2 + \tilde{b}^2 = -q^2 - \tilde{U} - \frac{2(1-q^2)}{\mathrm{cn}^2(i\omega_0)} + 2\tilde{u}_1^2, \tag{A.31}$$

$$\tilde{a}\tilde{b} \quad = i\, C^2 \left(\frac{\tilde{u}_0}{q^2} \frac{\mathrm{sn}(i\omega_0)\,\mathrm{dn}(i\omega_0)}{\mathrm{cn}^3(i\omega_0)} + \tilde{u}_1 \frac{\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)}{\mathrm{dn}^3(i\omega_1)} \right). \tag{A.32}$$

[21] In contrast to the $\mathbb{R} \times S^3$ case, the r.h.s. of (A.28) is not always real for arbitrary real values of $\tilde{\omega}_0$ and $\tilde{\omega}_1$. If $C^2 < 0$, we have to interchange η_0 and η_1 to obtain a solution properly normalized on AdS_3.

[22] Note that the Virasoro constraints require neither $a \geq b$ nor $a \leq b$. This means that both $\xi_1 = \exp\left(i\tilde{a}_0\tilde{T} + i\tilde{b}_0\tilde{X}\right)$ and $\exp\left(i\tilde{b}_0\tilde{T} + i\tilde{a}_0\tilde{X}\right)$ are consistent string solutions. It can be viewed as the $\tau \leftrightarrow \sigma$ transformation applied only to the $S^1 \subset S^5$ part while leaving the AdS_3 part intact.

The reality of $\tilde{a}$ and $\tilde{b}$ must also hold.

Since we are interested in closed string solutions, we should impose periodic boundary conditions. Let us define the period in the σ direction by

$$\Delta\sigma = \frac{2\mathbf{K}(k)\sqrt{1-v^2}}{\mu} = \frac{2q'\,\mathbf{K}(q)\sqrt{1-v^2}}{\mu} \equiv 2l \equiv \frac{2\pi}{n}\,, \tag{A.33}$$

which is equivalent to $\Delta\tilde{X} = 2\mathbf{K}(q)$ and $\Delta\tilde{T} = -2v\mathbf{K}(q)$. The periodicity conditions for the AdS variables are written as

$$\Delta t = 2\mathbf{K}(q)\left\{-iZ_2(i\tilde{\omega}_0) - v\tilde{u}_0\right\} + 2n'_{\rm time}\pi \equiv \frac{2\pi N_t}{n}\,, \tag{A.34}$$

$$\Delta\phi_1 = 2\mathbf{K}(q)\left\{-iZ_3(i\tilde{\omega}_1) - v\tilde{u}_1\right\} + \left(2n'_1+1\right)\pi \equiv \frac{2\pi N_{\phi_1}}{n}\,. \tag{A.35}$$

And from the periodicity in φ_1 direction, we have

$$N_{\varphi_1} = \mu\,\frac{\tilde{b}-v\tilde{a}}{\sqrt{1-v^2}} \in \mathbb{Z}\,. \tag{A.36}$$

We must further require the timelike winding N_t to be zero. Just as in the $\mathbb{R}\times S^3$ case, one can adjust the value of v to fulfill this requirement.[23] The integer $n'_{\rm time}$ is evaluated as

$$2n'_{\rm time}\pi = \frac{1}{2i}\int_{-\mathbf{K}}^{\mathbf{K}} d\tilde{X}\,\frac{\partial}{\partial\tilde{X}}\left[\log\left(\frac{\Theta_0(\tilde{X}-i\tilde{\omega}_0)}{\Theta_0(\tilde{X}+i\tilde{\omega}_0)}\right)\right]\,. \tag{A.37}$$

Then, by solving the equation $N_t = 0$, one finds an appropriate value of $v = v_t$. The absolute value of the worldsheet boost parameter v_t may possibly exceed one (the speed of light). In such cases, we have to perform the 2D transformation $\tau\leftrightarrow\sigma$ on the AdS space to get $v_t \mapsto -1/v_t$.

As usual, conserved charges are defined by

$$E \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{E} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\,\mathrm{Im}\left(\eta_0^*\,\partial_\tau\eta_0\right)\,, \tag{A.38}$$

$$S \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{S} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\,\mathrm{Im}\left(\eta_1^*\,\partial_\tau\eta_1\right)\,, \tag{A.39}$$

$$J \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{J} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\,\mathrm{Im}\left(\xi_1^*\,\partial_\tau\xi_1\right)\,. \tag{A.40}$$

which are evaluated as, for the current type (iii) case,

$$\mathcal{E} = \frac{nC^2\,\tilde{u}_0}{q^2(1-q^2)}\left[\mathbf{E} + (1-q^2)\left\{\frac{\mathrm{sn}^2(i\tilde{\omega}_0)}{\mathrm{cn}^2(i\tilde{\omega}_0)} - \frac{iv}{\tilde{u}_0}\,\frac{\mathrm{sn}(i\tilde{\omega}_0)\,\mathrm{dn}(i\tilde{\omega}_0)}{\mathrm{cn}^3(i\tilde{\omega}_0)}\right\}\mathbf{K}\right]\,, \tag{A.41}$$

$$\mathcal{S} = \frac{nC^2\,\tilde{u}_1}{q^2(1-q^2)}\left[\mathbf{E} - (1-q^2)\left\{\frac{1}{\mathrm{dn}^2(i\tilde{\omega}_1)} - \frac{ivq^2}{\tilde{u}_1}\,\frac{\mathrm{sn}(i\tilde{\omega}_1)\,\mathrm{cn}(i\tilde{\omega}_1)}{\mathrm{dn}^3(i\tilde{\omega}_1)}\right\}\mathbf{K}\right]\,, \tag{A.42}$$

$$\mathcal{J} = n\left(\tilde{a} - v\tilde{b}\right)\mathbf{K}\,. \tag{A.43}$$

[23]Note in $\mathbb{R}\times S^3$ case, the vanishing-N_t condition was trivially solved by $v = b/a$.

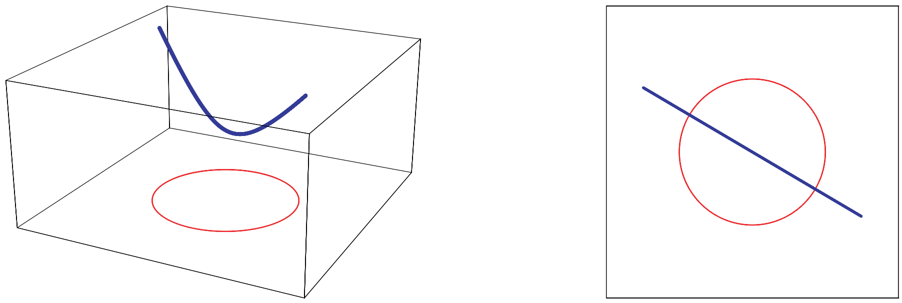

Figure 9: $\tilde{\omega}_{1,2} \to 0$ limit of type (iii) helical string becomes a folded string studied in [56].

It is interesting to see some of the limiting behaviors of this type (iii) helical string in detail.[24]

$\tilde{\omega}_{1,2} \to 0$ limit: folded strings on $AdS_3 \times S^1$. In the $\tilde{\omega}_{1,2} \to 0$ the timelike winding condition (A.34) requires $v = 0$, so the boosted worldsheet coordinates $(\tilde{T}, \tilde{X})$ become

$$(\tilde{T}, \tilde{X}) \to \left(\frac{\mu\tau}{q'}, \frac{\mu\sigma}{q'} \right) \equiv (\tilde{\mu}\tau, \tilde{\mu}\sigma) \equiv (\tilde{\tau}, \tilde{\sigma}) \,. \tag{A.44}$$

The periodicity condition (A.33) allows $\tilde{\mu}$ to take only a discrete set of values.

The profile of type (iii) strings now reduces to

$$\eta_0 = \frac{1}{\mathrm{dn}(\tilde{\sigma}, q)} \, e^{i\tilde{u}_0 \tilde{\tau}} \,, \qquad \eta_1 = \frac{q \, \mathrm{sn}(\tilde{\sigma}, q)}{\mathrm{dn}(\tilde{\sigma}, q)} \, e^{i\tilde{u}_1 \tilde{\tau}} \,, \qquad \xi_1 = \exp\left(i\sqrt{\tilde{U} - q^2} \; \tilde{\tau} \right) \,, \tag{A.45}$$

where $\tilde{u}_0^2 = \tilde{U}$ and $\tilde{u}_1^2 = \tilde{U} + 1 - q^2$. This solution is equivalent to T-transformation of (J_1, J_2) folded strings of [30], namely, (S, J) folded strings.[25] The conserved charges of (A.45) are computed as

$$\mathcal{E} = \frac{n\tilde{u}_0}{1 - q^2} \, \mathbf{E}(q) \,, \quad \mathcal{S} = \frac{n\tilde{u}_1}{1 - q^2} \Big(\mathbf{E}(q) - (1 - q^2) \mathbf{K}(q) \Big) \,, \quad \mathcal{J} = n\sqrt{\tilde{U} - q^2} \; \mathbf{K}(q) \,. \tag{A.46}$$

Rewriting these expressions in terms of the original imaginary modulus k, we find the following relations among conserved charges:

$$\left(\frac{\mathcal{J}}{\mathbf{K}(k)} \right)^2 - \left(\frac{\mathcal{E}}{\mathbf{E}(k)} \right)^2 = n^2 k^2 \,, \qquad \left(\frac{\mathcal{S}}{\mathbf{K}(k) - \mathbf{E}(k)} \right)^2 - \left(\frac{\mathcal{J}}{\mathbf{K}(k)} \right)^2 = n^2 (1 - k^2) \,, \tag{A.47}$$

as obtained in [57].

[24] It seems the original "spiky string" solution of [58] is also contained in the type (iii) class, although we have not been able to reproduce it analytically.

[25] Note the set, $\eta_{0,1} =$ the same as (A.45) and $\xi_1 = \exp[i\sqrt{\tilde{U} - q^2} \; \tilde{\sigma}]$, also gives a solution.

$q \to 1$ limit: logarithmic behavior. Another interesting limit is to send the elliptic modulus q to unity. In this limit, the spikes of the type (*iii*) string attach to the AdS boundary, and the energy E and AdS spin S become divergent. Again, the condition of vanishing timelike winding is fulfilled by $v = 0$, and the periodicity condition (A.33) implies that $\tilde{\mu}$ given in (A.44) goes to infinity. The profile becomes

$$\eta_0 = C \cosh(\tilde{\sigma} - i\tilde{\omega}_0)\, e^{i\tilde{u}_0 \tilde{\tau}} , \quad \eta_1 = C \sinh(\tilde{\sigma} - i\tilde{\omega}_1)\, e^{i\tilde{u}_1 \tilde{\tau}} , \quad \xi_1 = \exp\left(i\tilde{a}\tilde{\tau} + i\tilde{b}\tilde{\sigma}\right) , \tag{A.48}$$

where

$$C = \left(\cos^2\tilde{\omega}_1 - \sin^2\tilde{\omega}_0\right)^{-1/2} , \quad \tilde{u}_0^2 = \tilde{u}_1^2 = \tilde{U} . \tag{A.49}$$

The constants $\tilde{a}$ and $\tilde{b}$ satisfy the constraints

$$\tilde{a}^2 + \tilde{b}^2 = -1 + \tilde{U} , \qquad \tilde{a}\tilde{b} = C^2 \left(\tilde{u}_0 \sin\tilde{\omega}_0 \cos\tilde{\omega}_0 + \tilde{u}_1 \sin\tilde{\omega}_1 \cos\tilde{\omega}_1\right) . \tag{A.50}$$

The conserved charges are computed as

$$\mathcal{E} = nC^2\tilde{u}_0\Big(\Lambda - \sin^2\tilde{\omega}_0\,\mathbf{K}(1)\Big) , \quad \mathcal{S} = nC^2\tilde{u}_1\Big(\Lambda - \cos^2\tilde{\omega}_1\,\mathbf{K}(1)\Big) , \quad \mathcal{J} = n\tilde{a}\,\mathbf{K}(1) , \tag{A.51}$$

where we defined a cut-off $\Lambda \equiv 1/(1-q^2)$.

Let us pay special attention to the $\tilde{u}_0 = \tilde{u}_1 = \sqrt{\tilde{U}}$ case. For this case the energy-spin relation reads

$$\mathcal{E} - \mathcal{S} = n\sqrt{\tilde{U}}\;\mathbf{K}(1) . \tag{A.52}$$

Obviously the r.h.s. is divergent, and careful examination reveals it is logarithmic in $\mathcal{S}$. This can be seen by first noticing, on one hand, that the complete elliptic integral of the first kind $\mathbf{K}(q) \equiv \mathbf{K}(e^{-r})$ has asymptotic behavior

$$\mathbf{K}(e^{-r}) = -\frac{1}{2}\ln\left(\frac{r}{8}\right) + \mathcal{O}(r\ln r) , \tag{A.53}$$

while on the other, the degree of divergence for Λ is

$$\Lambda = \frac{1}{1-q^2} = \frac{1}{1-e^{-2r}} \sim \frac{1}{2r} , \qquad (\text{as } r \to 0) . \tag{A.54}$$

Since the most divergent part of $\mathcal{S}$ is governed by Λ rather than $\mathbf{K}(1)$, it follows that

$$\mathbf{K}(e^{-r}) \sim \mathbf{K}(1-r) \sim -\frac{1}{2}\ln\left(\frac{nC^2\tilde{u}_1}{16\mathcal{S}}\right) , \qquad (\text{as } r \to 0) , \tag{A.55}$$

at the leading order. Then it follows that

$$\mathcal{E} - \mathcal{S} \sim -\frac{n\sqrt{\tilde{U}}}{2}\ln\left(\frac{16\mathcal{S}}{nC^2\tilde{u}_1}\right) , \qquad (\text{as } r \to 0) , \tag{A.56}$$

as promised.

Let us consider the particular case $\tilde{U} = 1$, which is equivalent to $\tilde{a} = \tilde{b} = 0$ and $\tilde{\omega}_0 = -\tilde{\omega}_1$. The above dispersion relation (A.56) now reduces to

$$E - S = \frac{n\sqrt{\lambda}}{2\pi}\ln S , \tag{A.57}$$

omitting the finite part. This result was first obtained in [42] for the $n = 2$ case, and generalised to generic n case in [58].

One can also reproduce the double logarithm behavior of [56] (see also [57, 59−61]). To see this, let us set $\tilde{b} = 0$ and $\tilde{a} = \sqrt{\tilde{U} - 1}$, and rewrite the relation (A.52) as

$$\mathcal{E} - \mathcal{S} = \sqrt{\mathcal{J}^2 + n^2\,\mathbf{K}(1)^2} \sim \left[\mathcal{J}^2 + \frac{n^2}{4}\ln^2\left(\frac{2\mathcal{S}}{nC^2\sqrt{\tilde{U}}}\right)\right]^{1/2} . \tag{A.58}$$

There are two limits of special interest. The "slow long string" limit of [60], is reached by $\sqrt{U} \ll \lambda$, so that in the strong coupling regime $\lambda \gg 1$ the r.h.s. of (A.58) becomes

$$\mathcal{E} - \mathcal{S} \sim \sqrt{\mathcal{J}^2 + \frac{n^2}{4}\ln^2\mathcal{S}}\,. \tag{A.59}$$

Similarly, the "fast long string" of [60] is obtained by taking $\sqrt{U} \sim \lambda \gg 1$, resulting in

$$\mathcal{E} - \mathcal{S} \sim \left[\mathcal{J}^2 + \frac{n^2}{4}\left(\ln\left(\frac{\mathcal{S}}{\mathcal{J}}\right) + \ln\left(\ln r\right)\right)^2\right]^{1/2} \sim \sqrt{\mathcal{J}^2 + \frac{n^2}{4}\ln^2\left(\frac{\mathcal{S}}{\mathcal{J}}\right)}\,, \tag{A.60}$$

where we neglected a term $\ln(\ln r)$ which is relatively less divergent in the limit $r \to 0$.

A.2.2 Type (*iv*) helical strings

Let us finally present another AdS helical solution which incorporates the SL(2) "(dyonic) giant magnon" of [21, 29]. This solution, which we call the type (*iv*) string, is obtained by applying a shift $X \to X + i\mathbf{K}'(k)$ to the type (*i*) helical string. Its profile is given by

$$\eta_0 = \frac{C}{\sqrt{k}}\,\frac{\Theta_0(0)\,\Theta_0(X - i\omega_0)}{\Theta_0(i\omega_0)\,\Theta_1(X)}\,\exp\Big(Z_0(i\omega_0)X + iu_0T\Big)\,, \tag{A.61}$$

$$\eta_1 = \frac{C}{\sqrt{k}}\,\frac{\Theta_0(0)\,\Theta_3(X - i\omega_1)}{\Theta_2(i\omega_1)\,\Theta_1(X)}\,\exp\Big(Z_3(i\omega_1)X + iu_1T\Big)\,, \tag{A.62}$$

$$\xi_1 = \exp\left(iaT + ibX\right)\,. \tag{A.63}$$

We omit displaying all the constraints among the parameters (they can be obtained in a similar manner as in the type (*i*) case). The type (*iv*) solution corresponds to the helical-wave solution given in (A.18), and satisfy the string equations of motion of the form (A.19).[26]

$k \to 1$ limit: SL(2) "dyonic giant magnon". The SL(2) "dyonic giant magnon" is reproduced in the limit $k \to 1$, as

$$\eta_0 = \frac{\cosh(X - i\omega_0)}{\sinh X}\,e^{i(\tan\omega_0)X + iu_0T}\,, \quad \eta_1 = \frac{\cos\omega_0}{\sinh X}\,e^{iu_1T}\,, \quad \xi_1 = e^{\hat{a}T + i\hat{b}X}\,, \tag{A.64}$$

where

$$u_0^2 = u_1^2 + \frac{1}{\cos^2\omega_0}\,, \qquad (\hat{a}, \hat{b}) = (u_1, \tan\omega_0) \;\text{ or }\; (\tan\omega_0, u_1)\,. \tag{A.65}$$

[26] This can be easily checked by using a relation $1/k^2\,\mathrm{sn}^2(x,k) = \mathrm{sn}^2\left(x + i\mathbf{K}'(k), k\right)$.

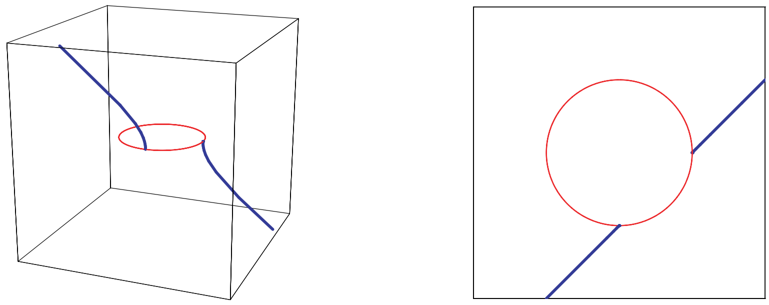

Figure 10: $k \to 1$ limit of type (iv) helical string ($\omega_0 = 0.785\,,\ u_0 = 1.41\,,\ u_1 = 0$): "giant magnon" solution in AdS space.

Due to the non-compactness of AdS space, the conserved charges are divergent. This is a UV divergence, and we regularise it by the following prescription. First change the integration range for the charges (see (A.38) - (A.40)) from $\int_0^{2l} d\sigma$ to $\int_\epsilon^{2l-\epsilon} d\sigma$, with $\epsilon > 0$, to obtain

$$\mathcal{E} = u_0 \cos^2 \omega_0 \left(\epsilon^{-1} - 1 \right) + \mathbf{K}(1)(u_0 - v \tan \omega_0)\,, \tag{A.66}$$

$$\mathcal{S} = u_1 \cos^2 \omega_0 \left(\epsilon^{-1} - 1 \right)\,, \tag{A.67}$$

$$\mathcal{J} = \mathbf{K}(1)(u_0 - v \tan \omega_0)\,, \tag{A.68}$$

then drop the terms proportional to ϵ^{-1} by hand. This prescription yields a regularised energy and an S^5 spin which are still IR divergent due to the non-compactness of the worldsheet. However, their difference becomes finite, leading to the energy-spin relation

$$(\mathcal{E} - \mathcal{J})_{\rm reg} = -\sqrt{(\mathcal{S})^2_{\rm reg} + \cos^2 \omega_0}\,. \tag{A.69}$$

Note that in view of the AdS/CFT correspondence, $\mathcal{E} - \mathcal{J}$ must be positive, which in turn implies $(\mathcal{E} - \mathcal{J})_{\rm reg}$ is negative.

Let us take $v = \tan \omega_0 / u_0$ in (A.64), and consider a rotating frame $\eta_0^{\rm new} = e^{-i\tilde{\tau}} \eta_0 \equiv \tilde{Y}_0 + i\tilde{Y}_5$. We then find $\tilde{Y}_5 = -i \sin \omega_0$ is independent of $\tilde{\tau}$ and $\tilde{\sigma}$, showing that the "shadow" of the SL(2) "dyonic giant magnon" projected onto the $\tilde{Y}_0$-$\tilde{Y}_5$ plane is just given by two semi-infinite straight lines on the same line. Namely, the shadow is obtained by removing a finite segment from an infinitely long line, where the two endpoints of the segment are on the unit circle $|\eta_0| = 1$ with angular difference $\Delta t = \pi - 2\omega_0$. Figure 10 shows the snapshot of the SL(2) "dyonic giant magnon", projected onto the plane spanned by $(\mathrm{Re}\,\eta_0, \mathrm{Im}\,\eta_0, |\eta_1|)$.

It is interesting to compare this situation with the usual giant magnon on $\mathbb{R} \times S^3$. In the sphere case, the "shadow" of the giant magnon is just a straight line segment connecting two endpoints on the equatorial circle $|\xi_1| = 1$. So the "shadows" of SU(2) and SL(2) giant magnons are just complementary. Using this picture of "shadows on the LLM plane", one can further discuss the "scattering" of two SL(2) "(dyonic) giant magnons" in the similar manner as in the SU(2) case.

These "shadow" pictures remind us of the corresponding finite-gap representations of both solutions, resulting from the SU(2) and SL(2) spin-chain analyses. While in the SU(2) case, a condensate cut, or a Bethe string, has finite length in the imaginary direction of the complex spectral parameter plane, for the SL(2) case, they are given by two semi-infinite lines in the same imaginary direction [21]. This complementary feature reflects the structural symmetry between the BDS parts of S-matrices, $S_{\mathrm{SU}(2)} = S^{-1}_{\mathrm{SL}(2)}$.

These "shadow" pictures also show up in matrix model context [24–27]. In a reduced matrix quantum mechanics setup obtained from $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$, a "string-bit" connecting eigenvalues of background matrices forming $\frac{1}{2}$-BPS circular droplet can be viewed as the shadow of the corresponding string. For the SU(2) sector, it is true even for the boundstate (bound "string-bits") case [26]. It would be interesting to investigate the SL(2) case along similar lines of thoughts.

B. Useful formulae

This appendix provides some formulae useful for computation involving Jacobi elliptic functions and elliptic integrals.

B.1 Elliptic functions and elliptic integrals near k = 1

The behavior of Jacobi elliptic functions around $k=1$ is discussed below.[27] We follow the method of [62], where they computed asymptotics around $k=0$.

Jacobi sn, cn and dn functions. The Jacobi sn function obeys an equation

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,. \tag{B.1}$$

Differentiating both sides with respect to k , one finds

$$\frac{\partial\,\mathrm{sn}(u,k)}{\partial k} = -\,\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)\int_0^{\mathrm{sn}(u,k)} \frac{k t^2\,dt}{\sqrt{1-t^2}\,(1-k^2t^2)^{3/2}}\,. \tag{B.2}$$

Taking the limit $k\to 1$ and substituting $u=i\omega$, we obtain

$$\left.\frac{\partial\,\mathrm{sn}(u,k)}{\partial k}\right|_{k\to 1} = \frac{i\,(\omega-\sin\omega\cos\omega)}{2\cos^2\omega}\,, \tag{B.3}$$

which is the first term in the expansion of the Jacobi sn function around $k=1$.

The asymptotics of the Jacobi cn and dn functions can be determined by the relations

$$\mathrm{sn}^2(u,k)+\,\mathrm{cn}^2(u,k)=1, \qquad \mathrm{dn}^2(u,k)+k^2\,\mathrm{sn}^2(u,k)=1\,. \tag{B.4}$$

[27] We make the elliptic moduli explicit in this section, and use the same conventions as [28].

Jacobi zeta function. The Jacobi zeta function behaves around $k = 1$ as

$$Z_0(u, k = e^{-r}) = \tanh u + \frac{z_2(u)}{\ln r} + r\, z_1(u) + \dots . \tag{B.5}$$

The functions $z_1(u)$ and $z_2(u)$ can be determined in the following way. The third term, $z_1(u)$, is calculated by the formula [63]:

$$\lim_{k \to 1} \mathbf{K}(k)\,(Z_0(u,k) - \tanh u) = -u\,, \tag{B.6}$$

while the second term, $z_2(u)$, can be determined by the relations

$$\frac{\partial Z_0(u,k)}{\partial u} = \mathrm{dn}^2(u,k) - \frac{\mathbf{E}(k)}{\mathbf{K}(k)}\,, \tag{B.7}$$

and

$$Z_0(u+v,k) - Z_0(u,k) - Z_0(v,k) = -k^2\,\mathrm{sn}(u,k)\,\mathrm{sn}(v,k)\,\mathrm{sn}(u+v,k)\,. \tag{B.8}$$

Complete elliptic integrals. For actual use of the relations (B.6) and (B.7), we need to know the asymptotics of complete ellitpic integrals. They are given by

$$\mathbf{K}(e^{-r}) = -\frac{1}{2}\ln r + \frac{3}{2}\ln 2 - \frac{1}{4} r \ln r + o(r \ln^m r)\,, \tag{B.9}$$

$$\mathbf{E}(e^{-r}) = 1 - \frac{1}{2} r \ln r + o(r \ln^m r)\,, \tag{B.10}$$

with $m > 1$. Changing the elliptic modulus from k to e^{-r}, the asymptotic behavior of elliptic functions around $r = 0$ are given by

$$\mathrm{sn}(i\omega, e^{-r}) = i \tan\omega - ir\,\frac{\omega - \sin\omega\cos\omega}{2\cos^2\omega} + O(r^2)\,, \tag{B.11}$$

$$\mathrm{cn}(i\omega, e^{-r}) = \frac{1}{\cos\omega} - r\frac{\omega\sin\omega - \sin^2\omega\cos\omega}{2\cos^2\omega} + O(r^2)\,, \tag{B.12}$$

$$\mathrm{dn}(i\omega, e^{-r}) = \frac{1}{\cos\omega} - r\frac{\omega\sin\omega + \sin^2\omega\cos\omega}{2\cos^2\omega} + O(r^2)\,, \tag{B.13}$$

$$Z_0(i\omega, e^{-r}) = i \tan\omega - ir\,\frac{\omega + \sin\omega\cos\omega}{2\cos^2\omega} + \frac{2i\omega}{\ln r} + O(r^2)\,. \tag{B.14}$$

B.2 Moduli transformations

We collect some formulae for $\mathrm{SL}(2,\mathbb{Z})$ transformations acting on elliptic functions.

Elliptic theta functions transform under the T-transformation as

$$\vartheta_0(z|\tau+1) = \vartheta_3(z|\tau)\,, \qquad \vartheta_1(z|\tau+1) = e^{\pi i/4}\,\vartheta_1(z|\tau)\,, \tag{B.15}$$

$$\vartheta_2(z|\tau+1) = e^{\pi i/4}\,\vartheta_2(z|\tau)\,, \qquad \vartheta_3(z|\tau+1) = \vartheta_0(z|\tau)\,, \tag{B.16}$$

and complete elliptic integrals with $q \geq 0$ transform as

$$\mathbf{K}(q) = k'\mathbf{K}(k)\,, \qquad \mathbf{K}'(q) = k'\left(\mathbf{K}'(k) - i\mathbf{K}(k)\right)\,, \qquad \mathbf{E}(q) = \mathbf{E}(k)/k'\,. \tag{B.17}$$

Jacobian theta functions, defined by

$$\Theta_\nu(z,k) \equiv \vartheta_\nu \left(\frac{z}{2\mathbf{K}(k)}, \tau = \frac{i\mathbf{K}'(k)}{\mathbf{K}(k)} \right), \quad (\nu = 0,1,2,3) \tag{B.18}$$

transform as

$$\Theta_0(z|\tau+1) = \Theta_3(z/k'|\tau), \qquad \Theta_1(z|\tau+1) = e^{\pi i/4}\,\Theta_1(z/k'|\tau), \tag{B.19}$$

$$\Theta_2(z|\tau+1) = e^{\pi i/4}\,\Theta_2(z/k'|\tau), \qquad \Theta_3(z|\tau+1) = \Theta_0(z/k'|\tau), \tag{B.20}$$

and Jacobian zeta functions defined by $Z_\nu(z,k) \equiv \partial_z \ln \Theta_\nu(z,k)$ transform as

$$Z_0(z|\tau+1) = Z_3(z/k'|\tau)/k', \qquad Z_1(z|\tau+1) = Z_1(z/k'|\tau)/k', \tag{B.21}$$

$$Z_2(z|\tau+1) = Z_2(z/k'|\tau)/k', \qquad Z_3(z|\tau+1) = Z_0(z/k'|\tau)/k'. \tag{B.22}$$

Therefore, the T-transformation acts on the elliptic modulus k as

$$q \equiv \left(\frac{\Theta_2(0|\tau+1)}{\Theta_3(0|\tau+1)} \right)^2 = i \left(\frac{\Theta_2(0|\tau)}{\Theta_0(0|\tau)} \right)^2 = \frac{ik}{k'}, \tag{B.23}$$

$$q' \equiv \left(\frac{\Theta_0(0|\tau+1)}{\Theta_3(0|\tau+1)} \right)^2 = \left(\frac{\Theta_3(0|\tau)}{\Theta_0(0|\tau)} \right)^2 = \frac{1}{k'}. \tag{B.24}$$

In terms of the modulus q defined in (A.23), the Jacobian sn, cn and dn functions are written as

$$\begin{aligned} \operatorname{sn}(z,q) &= k' \frac{\operatorname{sn}(z/k',k)}{\operatorname{dn}(z/k',k)}, \\ \operatorname{cn}(z,q) &= \frac{\operatorname{cn}(z/k',k)}{\operatorname{dn}(z/k',k)}, \\ \operatorname{dn}(z,q) &= \frac{1}{\operatorname{dn}(z/k',k)}. \end{aligned} \tag{B.25}$$

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[2] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[3] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[4] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[5] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [hep-th/0412188].

[6] N. Beisert and M. Staudacher, *Long-range $PSU(2,2|4)$ Bethe ansaetze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

[7] N. Beisert, *The* SU(2|2) *dynamic S-matrix*, hep-th/0511082.

[8] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, *Phys. Rev.* **D 73** (2006) 086006 [hep-th/0603038].

[9] B. Eden and M. Staudacher, *Integrability and transcendentality*, *J. Stat. Mech.* (2006) P11014 [hep-th/0603157].

[10] G. Arutyunov and S. Frolov, *On $AdS_5 \times S^5$ string S-matrix*, *Phys. Lett.* **B 639** (2006) 378 [hep-th/0604043].

[11] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [hep-th/0609044].

[12] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [hep-th/0610251].

[13] N. Dorey, D.M. Hofman and J.M. Maldacena, *On the singularities of the magnon S-matrix*, *Phys. Rev.* **D 76** (2007) 025011 [hep-th/0703104].

[14] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang-Mills*,*AIP Conf. Proc.* **646** (2003) 3.

[15] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys.* **A 39** (2006) 13095 [hep-th/0604135].

[16] H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, *JHEP* **09** (2006) 024 [hep-th/0605155].

[17] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, *J. Phys.* **A 39** (2006) 13119 [hep-th/0604175].

[18] H.-Y. Chen, N. Dorey and K. Okamura, *The asymptotic spectrum of the $N = 4$ super Yang-Mills spin chain*, *JHEP* **03** (2007) 005 [hep-th/0610295].

[19] H.-Y. Chen, N. Dorey and K. Okamura, *On the scattering of magnon boundstates*, *JHEP* **11** (2006) 035 [hep-th/0608047].

[20] . G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size effects from giant magnons*, *Nucl. Phys.* **B 778** (2007) 1 [hep-th/0606126];
D. Astolfi, V. Forini, G. Grignani and G.W. Semenoff, *Gauge invariant finite size spectrum of the giant magnon*, *Phys. Lett.* **B 651** (2007) 329 [hep-th/0702043].

[21] J.A. Minahan, A. Tirziu and A.A. Tseytlin, *Infinite spin limit of semiclassical string states*, *JHEP* **08** (2006) 049 [hep-th/0606145].

[22] M. Spradlin and A. Volovich, *Dressing the giant magnon*, *JHEP* **10** (2006) 012 [hep-th/0607009];
N.P. Bobev and R.C. Rashkov, *Multispin giant magnons*, *Phys. Rev.* **D 74** (2006) 046011 [hep-th/0607018];
M. Kruczenski, J. Russo, and A.A. Tseytlin, *Spiky strings and giant magnons on S^5*, *JHEP* **10** (2006) 002 [hep-th/0607044];
C. Kalousios, M. Spradlin and A. Volovich, *Dressing the giant magnon. II*, *JHEP* **03** (2007) 020 [hep-th/0611033];
S. Hirano, *Fat magnon*, *JHEP* **04** (2007) 010 [hep-th/0610027];
R. Roiban, *Magnon bound-state scattering in gauge and string theory*, *JHEP* **04** (2007) 048 [hep-th/0608049];

C.S. Chu, G. Georgiou and V.V. Khoze, *Magnons, classical strings and beta-deformations*, *JHEP* **11** (2006) 093 [hep-th/0606220].

[23] S. Ryang, *Wound and rotating strings in $AdS_5 \times S^5$*, *JHEP* **08** (2005) 047 [hep-th/0503239].

[24] D. Berenstein, D.H. Correa and S.E. Vazquez, *All loop BMN state energies from matrices*, *JHEP* **02** (2006) 048 [hep-th/0509015].

[25] S.E. Vazquez, *BPS condensates, matrix models and emergent string theory*, *JHEP* **01** (2007) 101 [hep-th/0607204].

[26] Y. Hatsuda and K. Okamura, *Emergent classical strings from matrix model*, *JHEP* **03** (2007) 077 [hep-th/0612269].

[27] D. Berenstein and S.E. Vazquez, *Giant magnon bound states from strongly coupled $N = 4$ SYM*, arXiv:0707.4669.

[28] K. Okamura and R. Suzuki, *A perspective on classical strings from complex sine-Gordon solitons*, *Phys. Rev.* **D 75** (2007) 046001 [hep-th/0609026].

[29] S. Ryang, *Three-spin giant magnons in $AdS_5 \times S^5$*, *JHEP* **12** (2006) 043 [hep-th/0610037].

[30] S. Frolov and A.A. Tseytlin, *Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors*, *Phys. Lett.* **B 570** (2003) 96 [hep-th/0306143].

[31] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[32] N. Gromov and P. Vieira, *The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve*, hep-th/0703191; *Constructing the AdS/CFT dressing factor*, hep-th/0703266.

[33] H.-Y. Chen, N. Dorey and R.F. Lima Matos, *Quantum scattering of giant magnons*, *JHEP* **09** (2007) 106 [arXiv:0707.0668].

[34] B. Vicedo, *Giant magnons and singular curves*, hep-th/0703180.

[35] N. Dorey and B. Vicedo, *On the dynamics of finite-gap solutions in classical string theory*, *JHEP* **07** (2006) 014 [hep-th/0601194].

[36] R. Ishizeki and M. Kruczenski, *Single spike solutions for strings on S^2 and S^3*, arXiv:0705.2429.

[37] A.E. Mosaffa and B. Safarzadeh, *Dual spikes: new spiky string solutions*, *JHEP* **08** (2007) 017 [arXiv:0705.3131].

[38] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys.* **B 671** (2003) 3 [hep-th/0307191].

[39] G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: new integrable system relations*, *Phys. Rev.* **D 69** (2004) 086009 [hep-th/0311004].

[40] K. Pohlmeyer, *Integrable hamiltonian systems and interactions through quadratic constraints*, *Commun. Math. Phys.* **46** (1976) 207;
F. Lund and T. Regge, *Unified approach to strings and vortices with soliton solutions*, *Phys. Rev.* **D 14** (1976) 1524;
F. Lund, *Note on the geometry of the nonlinear σ-model in two-dimensions*, *Phys. Rev.* **D 15** (1977) 1540.

JHEP11(2007)033

[41] A. Mikhailov, *An action variable of the sine-Gordon model*, *J. Geom. Phys.* **56** (2006) 2429 [hep-th/0504035]; *Baecklund transformations, energy shift and the plane wave limit*, hep-th/0507261; *A nonlocal poisson bracket of the sine-Gordon model*, hep-th/0511069.

[42] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[43] J.-Q. Liang and H.J.W. Müller-Kirsten, *Quantum tunneling for the sine-Gordon potential: energy band structure and boGomolny-Fateyev relation*, *Phys. Rev.* **D 51** (1995) 718.

[44] J.A. Minahan, *Circular semiclassical string solutions on $AdS_5 \times S^5$*, *Nucl. Phys.* **B 648** (2003) 203 [hep-th/0209047].

[45] J. Engquist, J.A. Minahan and K. Zarembo, *Yang-Mills duals for semiclassical strings on $AdS_5 \times S^5$*, *JHEP* **11** (2003) 063 [hep-th/0310188].

[46] M. Kruczenski and A.A. Tseytlin, *Semiclassical relativistic strings in S^5 and long coherent operators in $N = 4$ SYM theory*, *JHEP* **09** (2004) 038 [hep-th/0406189].

[47] J.A. Minahan, *Higher loops beyond the* SU(2) *sector*, *JHEP* **10** (2004) 053 [hep-th/0405243].

[48] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [hep-th/0306139].

[49] A. Rej, D. Serban and M. Staudacher, *Planar $N = 4$ gauge theory and the Hubbard model*, *JHEP* **03** (2006) 018 [hep-th/0512077].

[50] K. Zarembo, *Antiferromagnetic operators in $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 634** (2006) 552 [hep-th/0512079].

[51] R. Roiban, A. Tirziu and A.A. Tseytlin, *Slow-string limit and 'antiferromagnetic' state in AdS/CFT*, *Phys. Rev.* **D 73** (2006) 066003 [hep-th/0601074].

[52] A. Rej, M. Staudacher and S. Zieme, *Nesting and dressing*, *J. Stat. Mech.* (2007) P0808006 [hep-th/0702151].

[53] N. Dorey and B. Vicedo, *A symplectic structure for string theory on integrable backgrounds*, *JHEP* **03** (2007) 045 [hep-th/0606287].

[54] M. Kruczenski, *Spin chains and string theory*, *Phys. Rev. Lett.* **93** (2004) 161602 [hep-th/0311203].

[55] H. Dimov and R.C. Rashkov, *On the anatomy of multi-spin magnon and single spike string solutions*, arXiv:0709.4231.

[56] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *JHEP* **06** (2002) 007 [hep-th/0204226].

[57] N. Beisert, S. Frolov, M. Staudacher and A.A. Tseytlin, *Precision spectroscopy of AdS/CFT*, *JHEP* **10** (2003) 037 [hep-th/0308117].

[58] M. Kruczenski, *Spiky strings and single trace operators in gauge theories*, *JHEP* **08** (2005) 014 [hep-th/0410226].

[59] A.V. Belitsky, A.S. Gorsky and G.P. Korchemsky, *Logarithmic scaling in gauge/string correspondence*, *Nucl. Phys.* **B 748** (2006) 24 [hep-th/0601112].

[60] S. Frolov, A. Tirziu and A.A. Tseytlin, *Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT*, *Nucl. Phys.* **B 766** (2007) 232 [hep-th/0611269].

[61] P.Y. Casteill and C. Kristjansen, *The strong coupling limit of the scaling function from the quantum string Bethe ansatz*, *Nucl. Phys.* **B 785** (2007) 1 [arXiv:0705.0890].

[62] Y.K. Lin and F.A. Lee, *Expansions of Jacobian elliptic functions in powers of the modulus*, in *Technical Notes and Short Papers*, *J. Math. Comput.* **16** (1962) 372.

[63] P.F. Byrd and M.D. Friedman, *Handbook of elliptical integrals for engineers and physicists*, 2nd ed., vol. LXVII of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*, Springer, Berlin, Göttingen (1971).